A novel extension of the parallel-beam projection-slice theorem to divergent fan-beam and cone-beam projections

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The general goal of this paper is to extend the parallel-beam projection-slice theorem to divergent fan-beam and cone-beam projections without rebinning the divergent fan-beam and cone-beam projections into parallel-beam projections directly. The basic idea is to establish a novel link between the local Fourier transform of the projection data and the Fourier transform of the image object. Analogous to the two- and three-dimensional parallel-beam cases, the measured projection data are backprojected along the projection direction and then a local Fourier transform is taken for the backprojected data array. However, due to the loss of the shift invariance of the image object in a single view of the divergent-beam projections, the measured projection data is weighted by a distance dependent weight \( w(r) \) before the local Fourier transform is performed. The variable \( r \) in the weighting function \( w(r) \) is the distance from the backprojected point to the x-ray source position. It is shown that a special choice of the weighting function, \( w(r)=1/r \), will facilitate the calculations and a simple relation can be established between the Fourier transform of the image function and the local Fourier transform of the \( 1/r \)-weighted backprojection data array. Unlike the parallel-beam cases, a one-to-one correspondence does not exist for a local Fourier transform of the backprojected data array and a single line in the two-dimensional (2D) case or a single slice in the 3D case of the Fourier transform of the image function. However, the Fourier space of the image object can be built up after the local Fourier transforms of the \( 1/r \)-weighted backprojection data arrays are shifted and then summed in a laboratory frame. Thus the established relations Eq. (27) and Eq. (29) between the Fourier space of the image object and the Fourier transforms of the backprojected data arrays can be viewed as a generalized projection-slice theorem for divergent fan-beam and cone-beam projections. Once the Fourier space of the image function is built up, an inverse Fourier transform could be performed to reconstruct tomographic images from the divergent beam projections. Due to the linearity of the Fourier transform, an image reconstruction step can be performed either when the complete Fourier space is available or in parallel with the building of the Fourier space. Numerical simulations are performed to verify the generalized projection-slice theorem by using a disc phantom in the fan-beam case. © 2005 American Association of Physicists in Medicine. [DOI: 10.1118/1.1861792]

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I. INTRODUCTION

It is well known that the projection-slice theorem (PST) plays an important role in tomographic image reconstruction. The power of the PST lies in the fact that the Fourier transform of a single view of parallel-beam projections is mapped into a single line (two-dimensional case) or a single slice (three-dimensional case) in Fourier space via the PST. In other words, a complete Fourier space of the image object can be built up from the Fourier transforms of the sequentially measured parallel-beam projection data. Once all the Fourier information of the image object is known, an inverse Fourier transform can be performed to reconstruct the image. Along the direction of the parallel-beam projections there is a shift invariance of the image object in a single view of the parallel-beam projections. This is the fundamental reason for the one-to-one correspondence between the Fourier transform of parallel-beam projections and a straight line or a slice in the Fourier space. The name of the projection-slice theorem follows from this one-to-one correspondence.

In practice, divergent fan-beam and cone-beam scanning modes have the potential to allow for fast data acquisition. But image reconstruction from divergent-beam projections
poses a challenge. In particular, the PST is not directly applicable to divergent-beam projections since the shift invariance in a single view of projections is lost in the divergent-beam case. One way to bypass this problem is to explicitly rebin the measured divergent-beam projections into parallel-beam projections. This is one of the basic ideas used to solve the problem of fan-beam image reconstruction in computed tomography (CT). After rebinning, one can take advantage of fast Fourier transforms (FFT) to efficiently reconstruct images. There are some issues regarding the potential loss of the spatial resolution in the reconstructed images due to the data rebinning, but there are also some advantages in the noise distribution of a reconstructed image due to the nonlocal characteristic of the Fourier transform. Alternatively, a fan-beam projection can also be relabeled in terms of Radon variables so that the two-dimensional inverse Radon transform can be used to reconstruct images. In this way, a convolution-based fan-beam image reconstruction algorithm can be readily developed. It is referred to as the fan-beam filtered-backprojection (fan-beam FBP) algorithm. A potential disadvantage of the fan-beam FBP algorithm is that the weight in the backprojection step depends on the individual image pixels and thus the noise distribution may not be uniform throughout the image, which may pose problems in the clinical interpretation of tomographic images. In the case of cone-beam CT, it is much more complicated to rebin cone-beam projections into three-dimensional parallel-beam projections. The huge cone-beam data set also poses a challenge in terms of the potential data storage needed during the rebinning process. Alternatively, shift-invariant FBP algorithms have been developed to reconstruct images from cone-beam projection data set.2–11

Tomographic image reconstruction also appears in other x-ray imaging modalities such as tomosynthesis. In tomosynthesis, the acquired cone-beam projection data set is not complete in the sense of Tuy’s data sufficiency condition.3 Images are reconstructed using direct backprojection algorithms, FBP-based algorithms, or iterative algorithms.12 Several FBP-based image reconstruction algorithms have been proposed to reconstruct images from tomosynthetic projection data set.13–15 The difference among the FBP-based algorithms lies in the choice of filtering strategies in Fourier space. The qualities of the chosen filter directly affects the image quality in tomosynthesis.15 In current FBP-based tomosynthetic image reconstruction, the parallel-beam PST also plays a fundamental role. Although the projection data is acquired using a cone-beam geometry, the filter design is based on the assumption of parallel-beam geometry. Therefore, it is desirable to have a direct link between the Fourier space of an image object and the cone-beam projection data.

In this paper, a key observation is made about the PST: From a single view of parallel-beam projections, one may construct a partial Fourier space of an image object. A related question is posed for the divergent fan-beam and cone-beam projections: Is it possible to construct a partial Fourier space of an image object after one view of divergent-beam projections is acquired? The main objective of this paper is to develop a theoretical method of sequentially constructing the Fourier space of an image object during the acquisition of divergent-beam projections. A connection between the Fourier transform of the preweighted backprojection data array and the Fourier transform of the image object is developed. Analogous to the parallel-beam PST, the established relation is called a generalized projection-slice theorem (GPST) for divergent fan-beam and cone-beam projections.

This paper will be organized in the following manner. A review of the parallel-beam PST is given in Sec. II. The motivation is to formulate the well-known parallel-beam PST in a way that can be readily generalized to the divergent-beam case with a focus on the intuitive picture underlying the theorem. In Sec. III, a similar procedure that was utilized to develop the parallel-beam PST is extended to the divergent-beam projections. We refer this novel extension as a GPST. Numerical simulations of the GPST are implemented in the fan-beam case to verify the GPST by using a uniform disc phantom and a low-contrast Shepp-Logan phantom. These numerical results are presented in Sec. IV. Finally, some discussion of the GPST and conclusions are given in Sec. V.

II. BRIEF REVIEW OF CONVENTIONAL PARALLEL-BEAM PROJECTION SLICE THEOREM

For simplicity, suppose there exists an x-ray source that generates perfect parallel beams and the beams are wide enough to cover the whole image object as shown in Fig. 1. A detector is used to detect the attenuated x-ray beams. For a specific orientation \( \hat{n} \) of x-ray beams, a profile of the attenuated parallel x-ray beams is recorded. We call such a profile a single view of projections and denote it as \( g_{p}(\rho, \hat{n}) \). The subscript \( p \) is used to denote the fact that the projection data originated from the parallel x-ray beams. The Greek letter \( \rho \) is used to label the distance of a specific x-ray from the isoray as shown in Fig. 1.

If a function \( f(\hat{x}) \) is used to label the spatial distribution of the x-ray attenuation coefficients, the projection \( g_{p}(\rho, \hat{n}) \) is written as

\[
g_{p}(\rho, \hat{n}) = \int_{\mathbb{R}^2} d^2 \hat{x} f(\hat{x}) \delta(\rho - \hat{n} \cdot \hat{x}),
\]

(1)

where \( \hat{n} \) is a unit vector perpendicular to the unit vector \( \hat{n} \). In the two-dimensional case, Eq. (1) is a line integral along

![Fig. 1. Parallel beam x-ray projections and mathematical notations to label them.](image)
the line parallel to the unit vector $\hat{n}$ and the distance between a given integral line and the isoray is $\rho$. The definition (1) can also be viewed as a two-dimensional Radon transform in terms of another group of variables $(\rho, \hat{n}^\perp)$.

To connect the measured projections $g_p(\rho, \hat{n})$ to the Fourier components of the image function $f(\vec{x})$, a natural idea is to match the dimensions of the measured projections and the Fourier transform of the image object. Note that the measured projections are one dimensional for a two-dimensional image object. To compensate for the dimensional mismatch between measured projections and the dimensions of an image object, an operation called back projection is introduced. Namely, we put the measured projections back (back project) along the x-ray beams as shown in Fig. 2.

An important property of the backprojection operation in parallel beam projections is the shift invariance of the image object along the x-ray projection directions. Therefore, all the backprojection lines are equivalent. Thus, an equal weight should be assigned to each of the backprojection lines during the backprojection operation. The measured projection data are backprojected into two-dimensional data arrays such that the dimensional mismatch between the image object and its corresponding measured data disappears after the backprojection operation (Fig. 2).

If the result of the backprojection of a single view is denoted as $G_p(\vec{x}, \hat{n})$, then the backprojection operation may be expressed as

$$G_p(\vec{x}, \hat{n}) = g_p(\rho = \vec{x} \cdot \hat{n}^\perp, \hat{n}).$$

Now a connection between the Fourier transform of the image function $f(\vec{x})$ and the Fourier transform of backprojected data array $G_p(\vec{x}, \hat{n})$ may be established.

To do so, a local Fourier transform of the backprojected data array is defined as

$$F_p(\vec{k}, \hat{n}) = \int_{\mathbb{R}^2} d^2\vec{x} g_p(\vec{x} \cdot \hat{n}^\perp, \hat{n}) e^{-2\pi i \vec{k} \cdot \vec{x}}.$$  

In the second equality, the definition of backprojection Eq. (2) has been used. Remember that our objective is to connect the Fourier transform of an image object to the Fourier transform $[F_p(\vec{k}, \hat{n})]$ of the backprojected data array $G_p(\vec{x}, \hat{n})$. Thus it is natural to insert the definition of the projection $g_p(\rho, \hat{n})$ in Eq. (1) into Eq. (3) to obtain

$$F_p(\vec{k}, \hat{n}) = \int_{\mathbb{R}^2} d^2\vec{x} \delta_p(\vec{x} \cdot \hat{n}^\perp, \hat{n}) e^{-2\pi i \vec{k} \cdot \vec{x}}$$

$$= \int_{\mathbb{R}^2} d^2\vec{x} \int_{\mathbb{R}^2} d^2\vec{y} \delta(\vec{x} - \vec{y}) \delta(\vec{y} \cdot \hat{n}^\perp - \vec{r} \cdot \hat{n}^\perp) e^{-2\pi i \vec{k} \cdot \vec{x}}$$

$$= \int_{\mathbb{R}^2} d^2\vec{x} \delta(\vec{x} - \vec{r}) e^{-2\pi i \vec{k} \cdot \vec{x}}$$

$$\times \left\{ \int_{\mathbb{R}^2} d^2\vec{y} \delta(\vec{y} \cdot \hat{n}^\perp) e^{-2\pi i \vec{k} \cdot \vec{y}} \right\}.$$  

(4)

For convenience, a new vector $\vec{y}$ is introduced as (Fig. 3)

$$\vec{y} = \vec{x} - \vec{r} = y_p \hat{n} + y_\perp \hat{n}^\perp.$$  

(5)

In the second equality, the vector $\vec{y}$ is projected along the unit vector $\hat{n}$ and the transverse direction $\hat{n}^\perp$. The orthonormal vectors $\hat{n}$ and $\hat{n}^\perp$ define a local coordinate system. Using this decomposition, the integral in the curly bracket in Eq. (4) is calculated as

$$\int_{\mathbb{R}^2} d^2\vec{x} \delta(\vec{x} - \vec{r}) e^{-2\pi i \vec{k} \cdot \vec{x}}$$

$$= \int_{\mathbb{R}^2} d^2\vec{y} \delta(\vec{y} \cdot \hat{n}^\perp) e^{-2\pi i \vec{k} \cdot \vec{y}}$$

$$= \int_{-\infty}^{+\infty} dy_\perp \delta(y_\perp e^{-2\pi i \psi} (\vec{k}, \hat{n}))$$

$$= \delta(\vec{k} \cdot \hat{n}).$$  

(6)

Therefore, Eq. (4) simplifies to
and it is defined as

$$\tilde{f}(\vec{k}) = \int_{\Omega^2} d^2 \tilde{f}(\vec{r}) e^{-i2\pi \vec{k} \cdot \vec{r}}.$$  \hspace{1cm} (7)

Here $\tilde{f}(\vec{k})$ is the Fourier transform of the image function $f(\vec{x})$ and it is defined as

$$\tilde{f}(\vec{k}) = \int_{\Omega^2} d^2 \tilde{f}(\vec{x}) e^{-i2\pi \vec{k} \cdot \vec{x}}.$$  \hspace{1cm} (8)

Therefore, Eq. (7) gives the relation between the Fourier transform of the image object and the Fourier transform of the backprojected data array. The Dirac $\delta$ function in Eq. (7) dictates that the longitudinal components of the Fourier transform of back projected data array are zero. Namely, the Fourier transform of the back projected data array cannot generate nonzero Fourier components along the projection direction. In other words, the Fourier transform of the backprojected data array generates a line in the two-dimensional Fourier space of the image object. This is the well-known parallel beam projection-slice-theorem.\(^{16}\) Intuitively, this result is transparent if one remembers the nature of the parallel beam backprojection operation: The measured data has been backprojected along the x-ray projection direction with equal weights. Thus only the zero dc component of the Fourier transform is generated along the x-ray projection direction. All the nonzero components appear in a line or a slice perpendicular to the x-ray projection direction.

Suppose the parallel beam is continuously rotated around a fixed direction in an angular range $[0, 180^\circ]$. Then a complete Fourier space can be built up by using the Fourier transform of the backprojected data arrays. This is schematically illustrated in Fig. 4.

In the three-dimensional case, the parallel-beam PST may be illustrated similarly by performing equal weighted backprojection and taking a local Fourier transform. It also should be noted that there are many other ways to derive the parallel-beam PST. The motivation to revisit this theorem is to extract two key operations (backprojection and local Fourier transform) in building the Fourier space of the image object. In the next section, similar procedures will be employed in order to derive a generalized projection slice theorem for the fan-beam and cone-beam projections.

III. DIVERGENT BEAM PROJECTION SLICE THEOREM

In this section, an extension of the parallel-beam PST discussed in the preceding section will be established for divergent-beam projections. A divergent beam projection $g_d(\vec{r}, \vec{y})$ (for both fan beam and cone beam) is defined as

$$g_d(\vec{r}, \vec{y}(t)) = \int_{0}^{\infty} ds f[y(t) + s\vec{r}],$$  \hspace{1cm} (9)

where the source trajectory vector $\vec{y}(t)$ is parametrized by a parameter $t$ and $\vec{r}$ is a unit vector starting from the source position and pointing toward the image object (Fig. 5). The subscript $d$ is used to label the divergent-beam projections, while $p$ has been used to label the parallel-beam projections in Eq. (1). The image function $f(\vec{x})$ is assumed to have a compact support $\Omega$, i.e., it is nonzero only in a finite spatial region. Throughout the paper, a vector will be decomposed into a magnitude and a unit vector, e.g., $\vec{r} = r\hat{r}$. Obviously, the measured data in Eq. (9) has a one-dimensional structure in the fan-beam case (the detector is one dimensional). In the cone-beam case, the data has a two-dimensional structure (the detector is two dimensional).

The same strategy will also be employed to compensate for the dimensional mismatch between the projection data and image object as was used in the parallel-beam case. Namely, a backprojection operation shall be used to extend the measured fan-beam data into a two-dimensional data array. Likewise, a three-dimensional data array may also be generated from the measured cone-beam projection data.

When the measured divergent-beam projections are backprojected, a vital difference appears between the parallel-beam and divergent-beam projections. Namely, in a single view of divergent beam projections, the shift invariance of the image object is lost. This fact dictates that equal weighting is not appropriate for backprojecting the measured divergent-beam projections as it is in the parallel-beam cases. However, a striking feature of divergent-beam projec-
tions is their diverging nature. In other words, in each single view, all the backprojections converge at the same x-ray focal spot. Therefore, the backprojection operation is physically sensible only in a semi-infinite line: from the x-ray source position to infinity. Intuitively, an appropriate weight for the divergent-beam backprojection operation should be a function of the distance from the x-ray source position to the backprojected point. If the distance from an x-ray source position $\bar{y}(t)$ to a backprojected point $\bar{x}$ is denoted as $r$, i.e., $r=|\bar{x}-\bar{y}(t)|$, then a weighting function $w(r)$ can be assigned for backprojecting the divergent beam projections. Using this general form of weighting function, a weighted backprojection can be defined as

$$G_d(\bar{x},\bar{y}(t)) = w(r) = \frac{r}{|\bar{x}-\bar{y}(t)|} g_d(\bar{r},\bar{y}(t)) \bigg[ \bar{r} = \frac{\bar{x}-\bar{y}(t)}{|\bar{x}-\bar{y}(t)|}, \bar{y}(t) \bigg].$$  \hspace{1cm} (10)

A measured projection value $g_d(\bar{r},\bar{y})$ is multiplied by a weight $w(r=|\bar{x}-\bar{y}(t)|)$ and then backprojected along the direction $\bar{r}$ to a point $\bar{x}$ with distance $r=|\bar{x}-\bar{y}(t)|$ (Fig. 6).

After the backprojection operation in Eq. (10) is implemented, a one-dimensional fan-beam data array is converted into a two-dimensional data array. Similarly, a two-dimensional cone-beam data array is converted into a three-dimensional data array.

Analogous to the parallel beam case, the main objective of the present work is to establish a link between the Fourier transform of the above-defined backprojected divergent beam projections and the Fourier transform of the image function $f(x)$. In Eq. (10), the relevant variable in the backprojection operation is the distance $r$ and the orientation $\bar{r}$. They form a vector $\bar{r}$ as given below

$$\bar{r} = \frac{\bar{x}-\bar{y}(t)}{|\bar{x}-\bar{y}(t)|} r. \hspace{1cm} (11)$$

Thus it is convenient to take a Fourier transform of the backprojected data array in a local coordinate system centered at the x-ray focal spot. Namely, a Fourier transform of Eq. (10) with respect to the variable $\bar{r}$. To distinguish it from the parallel beam case, the Fourier transform of the backprojected data array is denoted as $\tilde{F}_d(\bar{k},\bar{y}(t))$. Here a vector $\bar{y}(t)$ explicitly labels the location where the local Fourier transforms are taken.

$$\tilde{F}_d(\bar{k},\bar{y}(t)) = \int_{\mathbb{R}^D} d^D\bar{r} g_d(\bar{r},\bar{y}(t)) e^{-i2\pi \bar{k} \cdot \bar{r}}.$$  \hspace{1cm} (12)

Substituting $g_d(\bar{r},\bar{y}(t))$ defined in Eq. (9) into Eq. (12), the following is obtained:

$$\tilde{F}_d(\bar{k},\bar{y}(t)) = \int_{\mathbb{R}^D} d^D\bar{r} g_d(\bar{r},\bar{y}(t)) e^{-i2\pi \bar{k} \cdot \bar{r}} = \int_{\mathbb{R}^D} d^D\bar{r} \left( \int_{0}^{+\infty} ds f(\bar{y}(t) + s\bar{r}) e^{-i2\pi \bar{k} \cdot \bar{r}} \right) e^{-i2\pi \bar{k} \cdot \bar{r}}.$$

$$= \int_{\mathbb{R}^D} d^D\bar{r} \left( \int_{0}^{+\infty} ds \left[ f(\bar{y}(t) + s'\bar{r}) \right] e^{-i2\pi \bar{k} \cdot \bar{r}} \right).$$

$$\hspace{1cm} (13)$$

In the last line, a new dummy variable $s'=s/r$ has been introduced and it will be written as $s$ again when there is no ambiguity. To establish the desired divergent-beam GPST, the following observation is found to be crucial: Both the vector $\bar{r}$ and its amplitude $r$ appear in the integrand and this complicates the problem. However, up to this point, the choice of the weighting function $w(r)$ is arbitrary. Therefore, a proper weighting function could be used to simplify the calculations in Eq. (13). An obvious choice is the following:

$$w(r) = \frac{1}{r}, \hspace{1cm} \text{i.e., } \hspace{1cm} w(r) = \frac{1}{r}. \hspace{1cm} (14)$$

After the above choice of the weighting function is made, the calculation of the Fourier transform of the backprojected divergent beam projections is significantly simplified to

$$\tilde{F}_d(\bar{k},\bar{y}(t)) = \int_{\mathbb{R}^D} d^D\bar{r} \left( \int_{0}^{+\infty} ds f(\bar{y}(t) + s\bar{r}) \right) e^{-i2\pi \bar{k} \cdot \bar{r}}.$$

$$= F \left\{ \frac{1}{s} f \left( \frac{\bar{k}}{s} \right) \right\}. \hspace{1cm} (15)$$

Here a symbol $F$ is introduced to label a Fourier transform. If the Fourier transform of the image function $f(x)$ is written as $\tilde{f}(\bar{k})$, then the following scaling property and shifting property of Fourier transform\(^{18}\) may be used to further simplify Eq. (15):

Scaling property (in $D$-dimensional space):

$$F\{f(s\bar{r})\} = \frac{1}{s^D} F \left( \frac{\bar{k}}{s} \right).$$  \hspace{1cm} (16)
Shifting property:
\[ F\{f(\bar{y}(t) + s\bar{t})\} = \frac{1}{s} \hat{f}\left(\frac{\bar{k}}{s}\right) \exp\left[i2\pi\bar{k}\cdot \bar{y}(t)\right]. \] (17)

Substituting Eq. (17) into Eq. (15) yields
\[ \tilde{F}_D(\bar{k}, \bar{y}(t)) = \int_{-\infty}^{\infty} ds d\bar{s} \hat{f}\left(\frac{\bar{k}}{s}\right) e^{i2\pi\bar{k}\cdot \bar{y}(t)/s}. \] (18)

To obtain a transparent and physical understanding of Eq. (18), the following change of variable is helpful:
\[ s' = \frac{k}{s} \quad \text{and} \quad ds = -k d(s'/s^2). \] (19)

In terms of the new variable of integration \( s' \), Eq. (18) may be expressed as
\[ \tilde{F}_D(\bar{k}, \bar{y}(t)) = \frac{1}{k^{D-1}} \int_{0}^{\infty} d(s') s'^{D-2} \hat{f}(s'\hat{k}) \exp[i2\pi s'\hat{k}\cdot \hat{y}(t)] \]
\[ = \frac{1}{k^{D-1}} \int_{0}^{\infty} ds d s'^{D-2} \hat{f}(s\hat{k}) \exp[i2\pi s\hat{k}\cdot \hat{y}(t)]. \] (20)

In the second line of Eq. (20), the integral variable has been written as \( s \) again since it is just a dummy variable of integration. A nice property of Eq. (20) is a decoupling of the radial part \((1/k^{D-1})\) from the angular part denoted as \( \tilde{C}_D(\hat{k}, \hat{y}(t)) \)
\[ \tilde{F}_D(\bar{k}, \bar{y}(t)) = \frac{1}{k^{D-1}} \tilde{C}_D(\hat{k}, \hat{y}(t)). \] (21)

\[ \tilde{C}_D(\hat{k}, \hat{y}(t)) = \int_{0}^{\infty} ds d s'^{D-2} \hat{f}(s\hat{k}) e^{i2\pi s\hat{k}\cdot \hat{y}(t)}. \] (22)

Equation (20), or equivalently Eqs. (21) and (22), gives the relation between the Fourier transform of the \( 1/r \)-weighted backprojection of a divergent data array and the Fourier transform \( \tilde{f}(\hat{k}) \) of the image function \( f(\bar{x}) \). However, due to the diverging nature of the beams, the information provided by a local Fourier transform of the backprojected data \( \tilde{C}_D(\hat{k}, \hat{y}(t)) \) does not simply correspond to a single slice or a single line in the Fourier space of the image object as in the parallel-beam case. However, this local Fourier transform is related to the desired Fourier transform of the image function in an elegant way. To see this point better, a composite variable \( p \) is introduced,
\[ p = \hat{k} \cdot \hat{y}(t). \] (23)

The meaning of the variable \( p \) is the projection distance of the x-ray source vector \( \hat{y}(t) \) on a specific orientation \( \hat{k} \) in the Fourier space. In terms of variable \( p \), the functions \( \tilde{F}_D(\bar{k}, \bar{y}(t)) \) and \( \tilde{C}_D(\hat{k}, \hat{y}(t)) \) may be rebinned into \( F_D(\bar{k}, p) \) and \( C_D(\hat{k}, p) \) respectively via the following relations:
\[ C_D(\hat{k}, p) = \tilde{C}_D(\hat{k}, \hat{y}(t)), \] (24)
\[ F_D(\bar{k}, p) = \tilde{F}_D(\bar{k}, \bar{y}(t)). \] (25)

Therefore, Eq. (22) can be recast into the following form:
\[ C_D(\hat{k}, p) = \int_{0}^{\infty} dp C_D(\hat{k}, p)e^{i2\pi kp}. \] (26)

In other words, \( C_D(\hat{k}, p) \) is linked to the Fourier transform \( \tilde{f}(\hat{k}) \) of the image object function by an inverse Fourier transform. A Fourier transform can be applied to obtain the Fourier transform \( \tilde{f}(\hat{k}) \) from the local Fourier transforms \( C_D(\hat{k}, p) \)
\[ = \int_{-\infty}^{\infty} dp F_D(\hat{k}, p)e^{-i2\pi kp}. \] (27)

Equations (21) and (25) have been used to achieve the second line of the above equation. Moreover, by noting the following fact:
\[ kp = \hat{k} \cdot \hat{y}(t) \] (28)

and the fact that factor \( \exp[-i2\pi \hat{k}\cdot \hat{y}(t)] \) is a phase shift, an intuitive understanding of Eq. (27) may be given as following: For each individual view of divergent-beam projections, a local Fourier transform of the \( 1/r \)-weighted backprojected data is performed.

However, the final objective is to build a global Fourier space of the image object. Thus the information of these local Fourier transforms has to be shifted into the same laboratory coordinate system (thus a phase factor \( \exp[-i2\pi \hat{k}\cdot \hat{y}(t)] \) is generated) and summed up to give a global Fourier space of the image object. This three-step procedure is illustrated in Figs. 7(a)–7(c) for the fan-beam case.

Thus we complete the derivation of our generalized projection-slice theorem (GPST) for the divergent-beam projections. It states that the Fourier transform of the image function is a sum of the shifted and rebinned local Fourier transforms of \( 1/r \) weighted fan-beam and cone-beam backprojection data.

Before the presentation of numerical simulations of the generalized projection slice theorem, a comment on Eq. (27) is given here. It is easy to see that the Fourier space reconstructed by Eq. (27) is intrinsically non-Cartesian. Thus the sampling density of the Fourier space is not uniform. The sampling density of the central Fourier space is higher than that of the peripheral Fourier space. This is similar to the projection data acquisition schemes in magnetic resonance imaging (MRI) in the two-dimensional case \(^{19,20}\) and in the three-dimensional case. \(^{17}\) To compensate for the nonuniform sampling in the Fourier space, a density weighting function \( 1/k^{D-1} \) should be used in transforming a non-Cartesian data set into a Cartesian data set. However, in the first line of Eq. (27), a prefactor of \( 1/k^{D-2} \) rather than \( 1/k^{D-1} \) appears and dominates the divergence of the Fourier space. To mitigate this discrepancy, a trick of integration by parts may be utilized to rewrite Eq. (27) as
In Eq. (29), the prefactor is $1/k^{D-1}$. Therefore, it represents a $1/k$ Fourier space sampling density compensation in the two-dimensional case and a $1/k^2$ Fourier space sampling density compensation in the three-dimensional case. Geometrically, it is also easy to understand this prefactor. It is essentially the inverse of the radial part of the Jacobian of a polar coordinate and spherical coordinate system in the two- and three-dimensional cases, respectively.

In order to understand how much projection data is required to construct a correct Fourier space for the image object, an inverse Fourier transform of Eq. (29) may be used to reconstruct an image point at $x$. The integral over variable $k$ along the radial direction may be carried out to yield a Dirac $\delta$ function: $\delta(\hat{k} \cdot (\vec{x} - \vec{y}(t)))$. This fact dictates the following data sufficiency condition for the construction of the Fourier space of an image object: in order to construct the Fourier space of an image point accurately, all the planes that pass through the image object should intersect the source trajectory at least once. This is essentially the well-known Smith data sufficiency condition.

Equations (27) and (29) are the central results of the present paper. Starting with the intrinsic nature of the diverging fan-beam and cone-beam projections, by a weighted backprojection, shifting and adding the local Fourier transforms, the Fourier space of the image object may be constructed. This theorem is valid for both fan beam and cone-beam projections.

IV. NUMERICAL SIMULATIONS AND RESULTS

The GPST Eqs. (27) and (29) have been theoretically derived for both fan-beam and cone-beam projections. In the derivation, the geometrical shape of the scanning path is arbitrary. For simplicity, in the remaining portion of the paper, numerical simulations are performed for the fan-beam case to validate the theoretical results.

IV.A. Phantoms

In the numerical simulations, two phantoms were utilized. One is a uniform disc phantom. The other one is a standard Shepp-Logan phantom.

The uniform disc phantom is defined by the following formula:

$$f(x, y, d) = \begin{cases} H \frac{1}{\sqrt{x^2 + y^2}} & \text{for } x^2 + y^2 \leq a, \\ 0 & \text{otherwise,} \end{cases}$$

where $a$ is the radius of the disc. The density of the disc has been set to unity. The Fourier transform of this phantom is the jinc function:

$$\mathcal{F}(k, \varphi) = \frac{2}{\pi a^2} \text{jinc}(2\pi ka) = 2 \pi a^{-2} J_1(2\pi ka) \frac{1}{2\pi ka}.$$
IV.B. Computation of the local Fourier transform

One of the key components in numerical computation of the Fourier space is to calculate the angular part \( \langle \tilde{C}(k, y) \rangle \) of the local Fourier transform \( \tilde{F}(k, y(t)) \). In the fan-beam case, the local Fourier transform is determined by a one-dimensional integral [Eq. (34)]. The details of the calculation are given in the Appendix and the final results are given here

\[
\tilde{C}(k, y(t)) = \text{Re} \{ \tilde{C}(k, y) \} + i \text{Im} \{ \tilde{C}(k, y) \},
\]

(32)

\[
\text{Re} \{ \tilde{C}(k, y) \} = \frac{1}{2\pi} g_{d} \int_{0}^{2\pi} d\varphi \frac{g_{d}(y(t), \hat{r})}{\hat{k} \cdot \hat{r}}.
\]

(33)

\[
\text{Im} \{ \tilde{C}(k, y) \} = -\frac{1}{2\pi} g_{d} \int_{0}^{2\pi} d\varphi \frac{g_{d}(y(t), \hat{r})}{\hat{k} \cdot \hat{r}}.
\]

(34)

Equations (32)–(34) have been used in Eq. (29) to reconstruct the Fourier space of the uniform disc phantom and the Shepp-Logan phantom. For a specific detector configuration such as equiangular and collinear equally spaced detector configurations, Eq. (34) may be further written as a shift-invariant filtering process.\(^{10}\)

IV.C. Coordinate system

For convenience, a circular source trajectory with radius \( R \) is used in the numerical simulations. The equiangular rays are used, and the coordinate system is illustrated in Fig. 8.

In this coordinate system, the vector \( \hat{y}(t) \) from origin to the source position is given by

\[
\hat{y}(t) = R(\cos t, \sin t).
\]

(35)

Thus the auxiliary variable \( p \) is given by

\[
p = \hat{k} \cdot \hat{y}(t) = R \cos(t - \varphi_k).
\]

(36)

Throughout the rest of the paper, the angular range \([0, 2\pi)\) of the scanning path is used so that the data redundancy is uniformly two. Namely, one value of the variable \( p \) corresponds to two possible view angles \( t_1 \) and \( t_2 \). Thus, a natural weight-scaling scheme to handle data redundancy is \( w = 1/2 \). When the data redundancy is not uniform such as what happened in the short scan case, an equal weight scheme may also be readily introduced in the data rebinning step.

IV.D. Procedure of numerical implementation

A data rebinning method has been used in numerical simulations. The main steps of the numerical implementation are summarized as follows:

1. For a given \( \varphi_k \) and a given distance \( p \in [-R, R] \), Eq. (36) is solved to obtain the corresponding view index \( t \in [0, 2\pi) \).

\[
t_p = \varphi_k + \arccos \frac{p}{R} \mod 2\pi.
\]

(37)

After the corresponding view index \( t_p \) is obtained, the real and imaginary parts of the local Fourier transform are calculated using formulas (33) and (34), respectively. The results are saved as \( \text{Re} \{ C_3(p, \varphi_k) \} \) and \( \text{Im} \{ C_3(p, \varphi_k) \} \).

2. Numerically calculate the derivative of \( \text{Re} \{ C_3(p, \varphi_k) \} \) and \( \text{Im} \{ C_3(p, \varphi_k) \} \) with respect to variable \( p \). The results are saved as \( \text{Re} \{ DC_3(p, \varphi_k) \} \) and \( \text{Im} \{ DC_3(p, \varphi_k) \} \).

3. Perform a one-dimensional FFT over the complex data array \( DC_3(p, \varphi_k) = \text{Re} \{ DC(p, \varphi_k) \} + i \text{Im} \{ DC(p, \varphi_k) \} \).

4. Scale the result of the one-dimensional FFT by the amplitude \( 2\pi k \) and separate the imaginary and real parts.

5. For different orientations in the Fourier space, repeat the above four steps.

IV.E. Results

All the implementation steps involve the relationship between a Fourier space and an image space, or a Fourier space and an auxiliary \( p \) space. Therefore, the familiar Nyquist-Shannon sampling theorem is utilized to guide the choice of the sampling rate. In the numerical simulations, the sampling rate of view angle is chosen as \( \Delta \varphi = 2\pi/1024 \). At the same time, the sampling rate of the detector is chosen as \( \Delta \gamma = \pi/(4 \times 512) \) with the total fan angle \( \pi/4 \).

Figures 9(a) and 9(b) show the reconstruction of the Fourier space for a disc phantom of two different sizes. When the size of the image object shrinks, the width of the first lobe in Fourier space becomes larger. Note that the imaginary part of the Fourier space of a uniform disc is zero.

Figures 10(a) and 10(b) show a quantitative comparison between the numerical result of the Fourier space radial profile with the mathematically exact result given by Eq. (29) at two different sizes of the image object.

Figure 11 shows the reconstructed image from the numerically calculated Fourier space (a) and the image intensity plot (b).

The same numerical procedure was also applied to a more realistic phantom, viz. Shepp-Logan phantom. Using the projection data generated from the mathematical phantom, a Fourier space for the Shepp-Logan phantom was computed. The real part is given in Fig. 12(a) and the imaginary part is
shown in Fig. 12(b). The inverse Fourier transform was used to reconstruct the image. The reconstructed image is shown in Fig. 12(c). A plot of image intensity along the horizontal central line is given in Fig. 12(d). Slight capping artifacts are visible in the reconstructed image in Fig. 12(c). In the image intensity plot Fig. 12(d), a small dc shift also presents. These artifacts are caused by a nonoptimal regularization scheme in the data filtering process.\(^1\) In other words, these artifacts are attributed to the numerical filtering process in calculating the local Fourier transform using Eq. (34), where a variant of the Hilbert filtering process is employed.

V. DISCUSSION AND CONCLUSIONS

V.A. Summary of the present work

The conventional projection slice theorem plays a pivotal role in tomographic image reconstruction from parallel-beam projections. It directly maps the Fourier transform of a specific view of the parallel-beam projection data into a radial line in the Fourier space of the image object in two-dimensional case. Sequential measurements at different view angles give a sequential construction of the Fourier space of the image object. The reconstruction of the image object can be performed after the complete Fourier space is constructed or, using the linearity of Fourier transforms, it can be sequentially performed after each projection is measured. In the present work, the above concept of reconstructing the Fourier space of an image object has been generalized to cases of fan-beam and cone-beam projections. The major results of this work are given in Eqs. (27) and (29). The key step in the derivation is to incorporate the divergent nature of the fan-beam and cone-beam projections into the backprojection procedure. This results in a \(1/r\)-weighting factor in the back-
projection. In contrast, the data is backprojected with an equal weight in the parallel-beam cases. After the back-projection step, a local Fourier transform is taken for the backprojected data array. Upon shifting and adding each of local Fourier transforms, the Fourier space of the image object is constructed. The difference between the parallel beam and the divergent beam is manifested in the shifting and adding steps. Due to the equal weighting of the parallel-beam backprojection operation, the shifting of the local Fourier transform does not induce an extra phase factor. However, in the divergent fan-beam and cone-beam cases, the shifting of a local Fourier transform into a common global laboratory frame induces an extra phase factor \( \exp[i2\pi \mathbf{k} \cdot \mathbf{y}(t)] = \exp(i2\pi kp) \). Thus, in the divergent fan-beam and cone-beam cases, adding all the local Fourier transforms together is equivalent to performing a Fourier transform with respect to the auxiliary variable \( p = \mathbf{k} \cdot \mathbf{y}(t) \). The physical meaning of variable \( p \) is the projection distance of the x-ray source position along a specific orientation \( \mathbf{k} \) in Fourier space. Although there are some intrinsic differences between parallel-beam and divergent-beam projections, the logical steps in the construction of a Fourier space of an image object are similar for both parallel-beam projections and divergent-beam projections. Thus the major results dictated by Eqs. (27) and (29) are dubbed as a generalized projection-slice theorem for divergent beam projections, although the concept of a line in the two-dimensional space or a slice in the three-dimensional space is not present for the divergent beam case.

V.B. Discussion of numerical implementations

In this work, a linear interpolation scheme is used in the computation of the local Fourier transform of the \( 1/r \)-weighted backprojection data array. Thus, deviation of the numerical results from the theoretical results is expected for high spatial frequencies in Fourier space. On the other hand, the range of auxiliary variable \( p \) is limited to \([−R,R]\), thus the integral in Eq. (27) or (29) is truncated. This fact causes limited resolution in the reconstructed Fourier space since the Fourier space resolution is determined by Nyquist sampling criterion by \( \Delta k = 1/2R \). Finally, we would also like to stress that there is a potential loss of spatial resolution due to interpolation process used for the data rebinning in the numerical simulations. In order to avoid the significant loss of the spatial resolution, one may develop a filtered back-projection method for the efficient reconstruction of the Fourier space to avoid the data rebinning procedure.

V.C. Future work

In the present paper, we proposed a new link between divergent beam projection data and the Fourier space of an image object. An inverse Fourier transform may be used to reconstruct images after the construction of the Fourier space. For complete source trajectories such as the well-known helical trajectory in CT, the authors do not claim that new GPST immediately provides an efficient and practical reconstruction algorithm. However, the GPST provides an intuitive understanding of how one can reconstruct images from measured divergent beam projections. In future work, we would like to study the relationship between the GPST and other known frameworks of image reconstruction. However, for imaging modalities such as tomosynthesis, the GPST provides a new way to extract information from the measured cone-beam projections. In the future filter design in tomosynthetic image reconstruction, one may directly start with cone-beam geometry, where the filling of the Fourier space is given by the new GPST in this paper. We are currently studying Fourier space reconstruction for tomosynthesis with a circular acquisition. It is computationally expensive to calculate the local Fourier transform for CT data acquisition where there are thousands views of projection data. But it will be less computationally expensive in the case of tomosynthesis where the number of view angles is reduced to approximately.
twenty. In addition, the linearity of the Fourier transform also provides an opportunity for parallel computation. Finally, an optimal filtering kernel will be developed to mitigate the image artifacts present in Fig. 12(c).

In conclusion, the well-known parallel-beam projection slice theorem has been generalized to the divergent fan-beam and cone-beam cases in the present paper. In order to fully exploit this new Fourier space method in tomographic image reconstruction, more comprehensive studies are needed to discover the tradeoffs among computational efficiency, image noise properties, and image resolution in the real clinical settings.

Fig. 12. Fourier space and the reconstructed image of a Shepp-Logan phantom. (a) The real part of the numerically calculated Fourier space. (b) The imaginary part of the numerically calculated Fourier space. (c) Reconstructed image from the Fourier space given by (a) and (b) using an inverse Fourier transform. (d) Intensity plot of along the horizontal central line. The display window for the reconstructed image in (c) is [0.95, 1.05].
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APPENDIX: TWO-DIMENSIONAL LOCAL FOURIER TRANSFORMS

Substituting the weighting function \( w(r) = 1/r \) into Eq. (12), the local Fourier transform, \( \tilde{F}_L[k, y(t)] \), is explicitly calculated as follows:

\[
\tilde{F}_L[k, y(t)] = \int_{\mathbb{R}^2} d^2 \tilde{r} G_d[x, y(t)] e^{-i2\pi \hat{k} \cdot \hat{r}}
\]

\[
= \int_{\mathbb{R}^2} d^2 \tilde{r} w(r) g_d[x, y(t), \tilde{r}] e^{-i2\pi \hat{k} \cdot \hat{r}}
\]

\[
= \int_0^\infty drw(r) \int_0^{2\pi} d\phi \cos[k \cdot y(t), \hat{r}] e^{-i2\pi \hat{k} \cdot \hat{r}}
\]

\[
= \int_0^{2\pi} d\phi \cos[k \cdot y(t), \hat{r}] \int_0^\infty dr f e^{-i2\pi \hat{k} \cdot \hat{r}}.
\]

Using the fact that

\[
\int_0^\infty dx e^{-i2\pi kx} = \int_{-\infty}^\infty dxu(x)e^{-i2\pi kx} = \frac{1}{2} \delta(k) + \frac{1}{2\pi i k}
\]

Equation (30) is calculated as

\[
\tilde{F}_L[k, y(t)] = \text{Re}[\tilde{F}_d[k, y]] + i \text{Im}[\tilde{F}_d[k, y]]
\]

\[
\text{Re}[\tilde{F}_L(k, y)] = \frac{1}{2} \int_0^{2\pi} d\phi \cos[k \cdot y(t), \hat{r}] \delta[k \cdot \hat{r}]
\]

\[
= \frac{1}{2} g_d[k \cdot y(t), \hat{r}] \delta[k \cdot \hat{r}]
\]

\[
\text{Im}[\tilde{F}_L(k, y)] = \frac{1}{2\pi i k} \int_0^{2\pi} d\phi \cos[k \cdot y(t), \hat{r}] \delta[k \cdot \hat{r}].
\]

Using the relation (21) between \( \tilde{F}_d[k, y(t)] \) and \( \tilde{C}_d[k, y(t)] \) in the fan beam case, function \( \tilde{C}_L[k, y(t)] \) is calculated as

\[
\tilde{C}_L[k, y(t)] = \text{Re}[\tilde{C}_d[k, y]] + i \text{Im}[\tilde{C}_d[k, y]]
\]

\[
\text{Re}[\tilde{C}_L(k, y)] = \frac{1}{2} g_d[k \cdot y(t), \hat{r}] \delta[k \cdot \hat{r}]
\]

\[
\text{Im}[\tilde{C}_L(k, y)] = -\frac{1}{2\pi} \int_0^{2\pi} d\phi \frac{g_d[k \cdot y(t), \hat{r}]}{k \cdot \hat{r}}. \quad (A8)
\]

These are the desired results as shown in Eqs. (32)–(34).

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\[^{q}\text{A. Maccoski, Medical Imaging Systems (Prentice-Hall, New Jersey, 1983).}\]


