AN ANALYSIS OF ELASTOGRAPHIC CONTRAST-TO-NOISE RATIO

TOMY VARGHESE and JONATHAN OPHIR
Ultrasonics Laboratory, Department of Radiology, The University of Texas Medical School, Houston, TX 77030 USA

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Abstract—We present a theoretical formalism and simulation results that allow the incorporation of the elastic contrast properties of tissues with simple geometries into the elastographic noise models developed previously. This analysis results in the computation of the elastographic contrast-to–noise ratio (CNR_e). The CNR_e in elastography is an important quantity that is related to the detectability of a lesion or inhomogeneity. In this paper, the upper bound on the elastographic CNR_e is derived for both a one-dimensional (1-D) and 2-D analytic plane-strain tissue model. The CNR_e in the elastogram depends on the contrast-transfer efficiency (CTE) for both the 1-D and 2-D geometries discussed in this paper. The 1-D model is used to characterize layered structures and the 2-D model is derived for circular inclusion within a background of uniform elasticity. A previously derived classical analytic solution of the elasticity equations, for a circular inclusion embedded in an infinite medium and subjected to a uniaxial compression, is used to compute the upper bound of the CNR_e. Monte Carlo simulations illustrate the close correspondence between the theoretical and simulation results. © 1998 World Federation for Ultrasound in Medicine & Biology.

Key Words: Ultrasound, Strain, Imaging, Contrast, Contrast-to–noise ratio, Cross-correlation, Elastography, Elastogram, Strain filter.

INTRODUCTION

Elastography is a method for imaging the elastic behavior of soft tissues, producing grey-scale strain images referred to as elastograms (Ophir et al. 1991, 1996, 1997, 1998). The process of generating an elastogram can be viewed as a two-step process, as illustrated in Fig. 1. First, the elastic medium is compressed to generate a strain field. The nature of the tissue compression, the elastic moduli of the medium, and the boundary conditions determine how the medium is strained in 3-D. Second, ultrasonic echo correlation techniques are used to estimate the axial strain field from the pre- and post-compression RF echo signals. Only the axial strain is estimated because the sampling of the RF signals along the beam axis is significantly finer than the lateral sampling interval.

In this paper, we combine the properties of the ultrasound imaging system and signal processing algorithms with the elastic behavior of the tissue, as illustrated in Fig. 1. This combined theoretical model enables prediction of the elastographic contrast-to–noise ratio (CNR_e) for layered or circular lesions embedded in a uniformly elastic background. The approach discussed in this paper may be modified in the future to include varying lesion sizes and shapes. However, the analysis in this paper is limited to sufficiently large diameters of hard and soft lesions.

We propose the use of analytical models to characterize tissue elastic properties. For layered structures, a 1-D model is proposed, where the strains in the tissue are directly proportional to their elastic moduli (Céspedes and Ophir 1993). For embedded circular lesions, a 2-D analytic solution of the elasticity equations (Honein and Herman 1933; Muskhelishvili 1963; Kallel et al. 1996) is used. In this case, the elastographic strains are not proportional to the elastic moduli, giving rise to fundamental limitations in displaying the modulus contrast. These limitations have been quantified in terms of the contrast-transfer efficiency (CTE) by Ponnekanti et al. (1995) and Kallel et al. (1996).

The CTE was defined as the ratio of the strain contrast as measured from the elastogram to the actual modulus contrast. The CTE for 1-D models equals 1 (maximum value) because the strain contrast and the
The noise performance of the strain estimator in elastography may be characterized using the strain filter (SF) developed by Varghese and Ophir (1997a). The novel concept in this paper is the incorporation of the analytic tissue CTE model into the SF formulation (see Fig. 1) for simple 1-D and 2-D tissue models. This results in a description of the combined effect of the elastic tissue parameters and the signal processing parameters; specifically, it provides a means of predicting the upper bound of the $CNR_e$ between layered tissues of various modulus contrasts, or between a lesion and its background.

Elastographic detectability of lesions depends on their size and contrast, the elastographic imaging system, display system and observer or detector performance (Belaid et al. 1994). The strain noise in the elastogram and the elastographic resolution are the other factors that limit the detector performance. Contrast detail curves for circular elastographic lesions with varying diameters and contrasts were analyzed by Belaid et al. (1994). They found that, for identical object contrasts, elastography had significantly higher detectability of all lesion diameters when compared to sonography.

The ability to make a decision on accepting or rejecting the presence of a lesion or layer is quantified by the contrast-to–noise ratio ($CNR_e$) which, in elastography, is computed from the means and variances of the strains in the lesion and the background, respectively (Bilgen and Insana 1997). The $CNR_e$ has been used as a basis for measuring the separability of two different data clusters. The $CNR_e$ is not rigorously related to the ideal observer performance; however, it is correlated with the visual impressions of the human observer when the lesions are larger than the noise correlations and the noise is roughly constant throughout the entire sonographic image (Insana and Hall 1994). Because the probability density function (pdf) for an homogeneous elastogram is characterized by Gaussian statistics (Belaid et al. 1994), similar results can be obtained for elastograms. This note does not discuss the performance of the ideal observer; rather, the strain dependence of the $CNR_e$ and the range of strains over which the $CNR_e$ is enhanced, are analyzed for some special cases.

The range of tissue modulus contrasts considered in this note was obtained from stiffness measurements in breast tissue in vitro (Krouskop et al. 1997). The stiffness measurements for normal breast tissue range from 20 ± 8 kPa for normal fat tissue ($n = 40$), 48 ± 15 kPa for normal glandular tissue ($n = 31$), to 220 ± 88 kPa for fibrous tissue ($n = 21$) using a 20% precompression and a 2%/s strain rate (Krouskop et al. 1997). For abnormal breast tissue, the stiffness measurements were 291 ± 67 kPa ($n = 23$) for ductal tumors and 558 ± 180 kPa ($n =
32) for infiltrating ductal carcinomas in breast tissue (Ophir et al. 1998) under similar loading conditions. The above measurements indicate that the stiffness ratio between the softest tissue in breast (normal fat) and hard lesions (carcinomas) is about 29 dB, and the ratio between ductal tumors and carcinomas is about 5.65 dB. Generally, lesions with a contrast of 6 dB and below may be defined as low-contrast lesions. The $CNR_e$ values that correspond to such low-contrast elastographic lesions are of primary interest in this note.

The analysis in this note does not account for the increase in the contrast in the elastograms due to the mechanical stress concentration artifacts (Céspedes 1993; Ponnekanti et al. 1995; Kallel and Ophir 1998). Contrast is defined as the ratio of the strain within the lesion to the strain in the homogeneous region in the absence of the lesion or far away from the lesion (Kallel et al. 1996). Defining the contrast in this manner underestimates the contrast observed in the elastogram. It has been noted that the presence of the mechanical stress concentration artifacts in elastography may improve the detectability of the lesion because it enhances the contrast between the lesion and the background region in its immediate vicinity. It has in fact been shown that the CTE improves for both hard and soft lesions when the peak artifacts are taken into account (Kallel and Ophir 1998). The upper bound computed in this note ignores these effects.

The following section presents the analytic tissue models used in this note. Monte Carlo simulations are used to corroborate the theoretically predicted $CNR_e$ results. Finally, the contributions of this note are summarized.

**THEORY**

The expression for the $CNR_e$ is derived by Bilgen and Insana (1997), and is given by:

\[
CNR_e = \frac{2(s_1 - s_2)^2}{\sigma_1^2 + \sigma_2^2},
\]

(1)

where $s_1 = s_f$ or $s_{L1}$ and $s_2 = s_B$ or $s_{L2}$ represent the mean value of strain in the lesion ($s_f$) or the first layer ($s_{L1}$) and the background ($s_B$) or second layer ($s_{L2}$), and $\sigma_1^2$, $\sigma_2^2$ denote the strain variances respectively. Note from eqn (1) that the $CNR_e$ in the elastogram can be estimated from knowledge of the means and the variances of the strain estimates within the lesion and the background or layers, respectively. The numerator in eqn (1) represents the strain contrast between the two layers or between the lesion and the background, and the denominator represents the average variance. A larger value of the $CNR_e$ represents better performance.

The $CNR_e$ in the elastogram depends on the contrast-transfer efficiency (CTE) for both the 1-D and 2-D geometries discussed in this paper. The CTE expressed in decibels can be written as:

\[
\eta(dB) = |C_o(dB)| - |C_o(dB)|,
\]

(2)

where $C_o$ is the strain contrast in the elastogram and $C_t$ is the modulus contrast. To depict the variation in the $CNR_e$ for a large range of modulus contrasts, we use a logarithmic scale (similar to the decibel scale) for the $CNR_e$, which is defined as follows:

\[
CNR_e(dB) = 20 \log_{10} \left[ \frac{2(s_1 - s_2)^2}{\sigma_1^2 + \sigma_2^2}; s_1 \neq s_2 \right].
\]

(3)

**Analytic solution for the 1-D tissue elasticity model**

For a 1-D model, the ratio of the strains in the individual layers is directly proportional to the ratio of their Young’s moduli. The 1-D model is applicable for modeling the strain in layered structures, as illustrated in Fig. 2a. The motion of each elastic tissue scatterer for a 1-D medium is modeled by considering an equivalent 1-D spring system (Céspedes and Ophir 1993). The spring constants are a function of the Young’s moduli of the tissue elements. Considering a two-layered medium, the ratio of the strains in the two layers can be written as follows:

\[
\frac{s_{L1}}{s_{L2}} = \frac{Y_{L1}}{Y_{L2}},
\]

(4)

where $Y_{L1}$ is the Young’s modulus of the tissue elements in the first layer and $Y_{L2}$ is the corresponding Young’s modulus in the second layer. The applied stress is assumed to propagate uniformly so that the localized stress is constant throughout the medium of uniform elasticity. Equation (3) assumes that both the layers have the same value of the Poisson’s ratio. The straight line in Fig. 3 illustrates the relationship between the modulus contrast and the strain contrast in elastography for the 1-D model.

**Analytic solution for a 2-D elasticity model**

A 2-D or 3-D model is necessary for the analysis of the contrast detectibility of lesions in elastography to account for the fundamental limitation on the CTE, first observed by Ponnekanti et al. (1995) using finite element analysis (FEA). We use the analytical solution of the 2-D elasticity problem for a circular lesion embedded in a homogeneous infinite medium (Honein and Herman 1933; Muskhelishvili 1963), to obtain the strains within the lesion and the background, under the assumption of a plane-strain state. A complete theoretical analysis of the
stresses and strains for this model is presented by Kallel et al. (1996). Closed-form expressions for the strains within the circular lesion and in the background and the spatial variation in the strains over the entire analytic phantom have been derived (Kallel et al. 1996), for the simple geometry illustrated in Fig. 2b. The 2-D plane illustrated in Fig. 2b is the cross-section of a cylindrical lesion with radius \( r \) and shear modulus \( \mu_I \) embedded in an infinite medium with shear modulus \( \mu_B \), and subjected to a uniform uniaxial compression \( F \).

Equations (5) and (6) provide the strain value within the lesion and the background only. The strain contrast in the elastogram for the 2-D analytic model is given by:

\[
P = \frac{(1 - \nu) F}{2(1 + \nu)}
\]  

where \( \nu \) is the Poisson’s ratio, \( F \) is the applied uniform uniaxial force, \( \mu_I \) and \( \mu_B \) are the shear moduli in the lesion and the background, respectively. The shear modulus is related to the Young’s modulus (\( E \)) in the plane-strain state by:

\[
\mu = \frac{E}{2(1 + \nu)}.
\]  

Substituting eqn (7) into the expression for the modulus contrast, we obtain the ratio of the Young’s moduli in the lesion to that in the background. A plot of the strain contrast vs. the modulus contrast (contrast-transfer characteristic) is shown in Fig. 3, illustrating the reduction in the strain contrast in the elastogram for soft lesions. For the hard lesions, however, the strain contrast is close to the modulus contrast (Ponnekanti et al. 1995).
Estimating the contrast-to–noise ratio using the strain filter

The behavior of the upper bound of the elastographic SNR\textsubscript{e} with tissue strain is defined as the SF (Varghese and Ophir 1997a), and is given by:

\[ \text{SNR}_{e}^{UB} = \frac{s}{\sigma(s)\text{ZZLB}_p}, \]  

where \( s \) is the tissue strain, and \( \sigma(s)\text{ZZLB}_p \) is the modified Ziv-Zakai lower bound (ZZLB) (Weinstein and Weiss 1984) on the standard deviation of the strain estimator (Varghese and Ophir 1997a). The lower bounds on the strain estimation variance are denoted with an additional subscript \( p \) to illustrate that these variances are computed for partially correlated signals. These lower bounds converge to the classical bounds when \( p = 1 \). The SF plotted in Fig. 4 shows that the measurement process allows only a selected range of strains to be displayed on the elastogram with a reasonable SNR\textsubscript{e} and, consequently, has bandpass characteristics in the strain domain. Because the performance of the strain estimator in elastography depends on the tissue strain (see Fig. 4), the contrast and, subsequently, the \( \text{CNR}_{e} \) depend on all the factors that influence the SF (Varghese and Ophir 1997a; Varghese et al. 1998). Figure 4 illustrates the process of estimating \( \text{CNR}_{e} \) between two different tissue strain values using the SF. For the two different strain values, we obtain the corresponding SNR\textsubscript{e} and, thereby, the variance values from the SF. An upper bound on the \( \text{CNR}_{e} \) is obtained using eqn (1) because the SF formulation is an upper bound.

Analysis of the contrast-to–noise ratio using the elasticity models

An analytical solution for the variation in the \( \text{CNR}_{e} \) can now be obtained for the specific 1-D and 2-D geometries by incorporating the strains obtained using the above elasticity models into the SF. In general, we can obtain the upper bound on the \( \text{CNR}_{e} \) for any geometry, provided the analytic solution is available. We use the following parameter values for the calculations, unless stated otherwise: \( s = 1\% \), window length (\( Z \)) = 2 mm, fractional bandwidth (\( B \)) = 0.6, spatial center frequency (\( k_o \)) = 30.6 mm\(^{-1}\) (7.5-MHz center frequency) and overlap factor (\( \Delta z \)) = \( Z/2 \) (i.e., 50% window overlap).

Figure 5 compares the behavior of the upper bound of the \( \text{CNR}_{e} \) using both 1-D and 2-D models for the geometries illustrated in Fig. 3. For the 1-D model, the strain is modeled by considering an equivalent 1-D spring system (Céspedes 1993), and the 2-D analytic solution is used for the 2-D model (Kallel et al. 1996). Note that, for soft lesions, the increased signal decorrelation that occurs at large contrasts using the 1-D model drastically reduces the \( \text{CNR}_{e} \). However, due to the geometry of the 2-D model (soft lesion within a hard background), the strains within the lesion are quite small even at large contrasts. For hard lesions, both the 1-D and 2-D model provide similar results. Using the 2-D model, we observe a difference of about 10 dB in the \( \text{CNR}_{e} \).
between the soft lesion and the hard background. Note that the suboptimal CTE for soft lesions in the 2-D model is responsible for the improvement of the \( CNR_e \) when compared to the 1-D model. The \( CNR_e \) is improved because, in the 2-D geometry, the strains in the soft lesion are lower than for the corresponding 1-D model.

The behavior of the upper bound of the \( CNR_e \) at different values of the background tissue strain is illustrated in Fig. 6 for the 2-D analytic tissue model. Note that, with increased strain, the \( CNR_e \) for the soft lesions reduces at high contrasts, and exhibits a behavior similar to the 1-D model. This deterioration in the performance for the soft lesion is due to the increased decorrelation with larger strains. The strain-estimation variance decreases with an increase in the center frequency and the bandwidth, thereby improving the \( CNR_e \).

The \( CNR_e \) performance with different window lengths with a background strain of \( s = 3\% \) is illustrated in Fig. 7. Note that the \( CNR_e \) for the hard lesions increases with an increase in the window length; however, for the soft lesion, we observe the opposite effect. The above result is due to the presence of a window length for each strain value where the strain-estimation variance is minimized (Varghese et al. 1998). Large windows provide low variances for small strains, and smaller windows are essential for accurate estimation of large tissue strains to minimize decorrelation within the window (Céspedes and Ophir 1993; Varghese et al. 1998). In Fig. 7, the soft lesion is strained more and, therefore, we obtain the best performance with the smallest window length. Similarly, because the hard lesion undergoes less strain, a larger window provides better performance.

The behavior of the upper bound of the \( CNR_e \) with the background tissue strain and modulus contrast results in a contrast contour map that is presented in Fig. 8a for the 1-D model and in Fig. 8b for the 2-D model. For hard lesions, the \( CNR_e \) performance is similar for both the 1-D and 2-D models, and increases with the modulus contrast for strains in the range from \(-60\) dB to \(-30\) dB. With smaller applied strains, the \( CNR_e \) in the elastograms reduces slowly and is limited by the sonographic SNR. However, for large applied strains (\( \geq -30\) dB), we observe an abrupt drop in the \( CNR_e \) as predicted by the SF, due to increased signal decorrelation effects. The contrast contour maps for the soft lesions mirror the behavior observed with the hard lesions for the 2-D model, albeit at a lower value of the \( CNR_e \). However, for the 1-D model, the maximum value of the \( CNR_e \) shifts towards higher modulus contrasts with a reduction in the background strain. Therefore, to obtain high \( CNR_e \) for layers with a large modulus contrast, the applied compression (background strain) has to be small. In addition, observe that we obtain the same maximum value of the \( CNR_e \) for both the hard and soft lesions with the 1-D model, because the CTE equals 1. However, for the 2-D model, because the CTE is always less than 1 for the soft lesions, we observe a 10-dB reduction in the \( CNR_e \) with the 2-D model.

Note, from Fig. 8 and eqn (1), that the highest values of the \( CNR_e \) are obtained where two conditions are satisfied; first, the differences in mean strain values...
Fig. 8. Contrast contour maps, illustrating the variation in the upper bound of the \( \text{CNR}_w \) with the modulus contrast with varying background strains for both (a) the 1-D and (b) 2-D tissue elasticity models. The curves were obtained using the following parameters: \( k_v = 30.6 \text{ mm}^{-1} \); \( B = 0.6; Z = 2 \text{ mm} \) with a 50\% overlap in the data segments.
must be large, and second, the sum of the variances of the strain estimates should be small. The improvement of the 
$CNRe$ at low modulus contrasts is primarily due to the small strain variances and, at high modulus contrasts, the improvement in the 
$CNRe$ is due to the large difference in the mean strains. Note, from the contrast contour map in Fig. 8, that, when the differences in the mean strain values are small (region around the middle of the graph at low contrasts), the $CNRe$ value obtained is almost zero. In addition, the regions with large strains (corresponding to large variances in the strain estimate due to signal decorrelation) also contribute to low $CNRe$ values. Each contour in Fig. 8 corresponds to an increase in the $CNRe$ by a factor of about 11 dB. The maximum value of the $CNRe$ is obtained for strains around $–40$ dB. The contrast contour maps in Fig. 8 were generated for the specific set of system- and signal-processing parameters discussed earlier, and will vary with the choice of parameters.

**SIMULATION RESULTS**

Monte–Carlo simulations in MATLAB are used in this section to corroborate the theoretical results presented in the previous section for the 1-D and 2-D tissue models. The B-scans for the 1-D and 2-D analytic models were obtained as follows: First, the analytic model was used to generate displacement information with applied strain. An acoustic Rayleigh scattering model was added to the displacement information to simulate tissue scattering profiles. Pre- and postcompression RF signals were generated using a convolutional model and a simulated acoustic transducer, with a 7.5-MHz center frequency and 60% bandwidth. A detailed description of the B-scan simulator is presented in Kallel et al. (1996). The RF signals were digitized using a 48-MHz sampling frequency. The speed of sound in tissue is assumed to be constant at 1540 m/s. The transducer point-spread function (PSF) was simulated using a 1-D Gaussian modulated cosine pulse.

Elastograms are generated by applying normalized cross-correlation techniques to congruent pairs of the RF pre- and postcompression signals. The ideal strain images obtained using the analytic solution and the elastograms obtained using simulations are illustrated in Fig. 9a–d. The 1-D model consists of two layers with a 20-dB contrast between the layers, as observed in Fig. 9a and b. The ideal elastogram, obtained using eqn (4), is illustrated in Fig. 9a and the estimated elastogram is shown in Fig. 9b. For the 2-D analytic tissue model, the diameter of the lesion in the $40 \times 40$ mm phantom was $17$ mm with a 20-dB modulus contrast. The ideal strain images in Fig. 9c and the estimated elastogram in Fig. 9d for the 2-D model clearly indicate the presence of the stress concentration (mechanical) and shadowing artifacts around the lesion (Céspedes 1993). The shadowing artifacts in the analytic model are discussed in detail by Kallel et al. (1996). The $CNRe$ in the elastogram is computed between the layers for the 1-D model and, for the 2-D model, between a small region in the lesion at the center of the phantom and small regions at the four corners of the phantom, to reduce the impact of the stress concentration artifacts on the $CNRe$. The median value from the four estimates of the $CNRe$ is selected. The variation in this median value of the $CNRe$ over 25 independent simulations at different contrasts, is used to corroborate the theoretical prediction for the upper bound of the $CNRe$.

Figure 10a, b presents results for the $CNRe$ obtained using the 1-D and 2-D analytic models. The graphs with the error bars and symbols denote the simulation results: “o” for $s = 1\%$. The contrast between the layers and between the lesion and the background is varied from $–30$ dB to $+30$ dB. The solid curves denote the theoretical value of the $CNRe$ in Fig. 10, which forms the upper bound on the simulation results. In addition, observe that the simulation and theoretical results follow the same trend.

**DISCUSSION AND CONCLUSIONS**

In this note, we derived the upper bound on the $CNRe$ for specific 1-D and 2-D analytical models. Simulation results illustrate the close correspondence between the theoretical and simulation results. Knowledge of the theoretical upper bound on the $CNRe$ in elastography is crucial for determining the ability to discriminate between different regions in the elastograms. Trade-offs among the various elastographic parameters, namely the elastographic $SNRe$, sensitivity, dynamic range, $CNRe$ and resolution can be analyzed using the methods discussed in this paper.

The $CNRe$ has been predicted for the two specific 1-D and 2-D geometries and tissue elasticity models. The CTE for both the 1-D and 2-D elasticity models and the elastographic noise characterized by the SF determine the $CNRe$ in elastography. The reduction in the $CNRe$ for soft lesions due to the lower value of the CTE for the 2-D tissue model is observed both in the theoretical and simulation results. However, this effect also allows a larger range of compression to be applied for a 2-D geometry before the onset of signal decorrelation, as observed from the contrast contour maps in Fig. 8. It has to be noted that, with increased signal decorrelation, softer regions in the elastogram may become more detectable to the human observer due to their increased variance. However, the corresponding $CNRe$ in the elastogram will be low due to the increased strain variance.
This note does not consider the improvement in detectability occurring in such cases.

The strain dependence of the elastographic $CNR_e$ is illustrated here. The $CNR_e$ in the elastograms depends to a large extent on the strain induced in tissue due to the applied compression. Incorporation of the elastic tissue models into the theoretical analysis of the elastographic imaging process provides a means of studying the upper bound on the elastographic $CNR_e$, which is dependent on the elastic tissue model as well as on the ultrasonic instrumentation parameters. This approach can be used to predict the upper bound on the $CNR_e$, provided an analytic or simulated solution for the strains for a specified geometry is available. The range of applied compressions over which we obtained the desired elastographic $CNR_e$ is illustrated using the contrast contour maps. The contrast contour maps provide a means of maximizing the $CNR_e$ in the elastogram for the given ultrasound system and signal-processing parameters. The contrast contour maps, therefore, illustrate the range of strains and contrasts over which the $CNR_e$ and, thereby, the detectability of the lesion or layer in the elastogram is enhanced.

The analysis in this note does not account for the increase in the object contrast, especially for high-contrast situations arising from mechanical stress concentration artifacts (Céspedes 1993; Ponnekanti et al. 1995; Kallel et al. 1996; Kallel and Ophir 1998). However, because the contribution of the stress concentrations increases with lesion contrast, the $CNR_e$ provides a good indication of the performance of the strain estimator only at low contrasts. Detectability of low-contrast lesions is of particular interest in elastography (Belaid et al. 1994) because the high-contrast lesions can generally be detected in the presence of the stress concentrations. In addition, this analysis of the $CNR_e$ was obtained at a
fixed depth in tissue (at the focus of the ultrasonic beam) and by ignoring the effects of lateral and elevational decorrelation. However, we may obtain a family of CNR curves and the contrast contour maps at different depths or at different lateral and elevational positions by properly derating the SF (Varghese and Ophir 1997b; Kallel et al. 1997). The CNR can, therefore, be predicted for lesions of varying modulus contrasts that reside at different axial and lateral positions in the elastogram.

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