A GENERAL SOLUTION FOR CATHETER POSITION EFFECTS FOR STRAIN ESTIMATION IN INTRAVASCULAR ELASTOGRAPHY

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Abstract—Intravascular ultrasound (US) elastography reveals the elastic properties of vascular tissue and plaque. However, misalignment of the US catheter in the vessel lumen can cause incorrect strain estimation in intravascular US elastography caused by strain projection artifacts. In this paper, we present a general theoretical solution where the impact of catheter eccentricity, tilt and noncoplanar errors on the strain estimates are derived. Appropriate corrections to strain estimates can then be applied with prior knowledge of the catheter position information to reduce the strain projection artifacts. Simulations using a frequency-domain-based algorithm that models intravascular US imaging before and after a specified deformation are presented. The simulations are used to verify the theoretical derivations for two displacement situations (linear and nonlinear) under intraluminal pressure, with and without stress decay. The linear displacement case demonstrates that the correction factor is dependent only on the angle between the US beam and the cross-sectional plane of the vessel. For the nonlinear displacement case, where a \( \frac{l}{r} \) stress decay in the displacement is modeled, the correction factor becomes a more complicated function of the azimuthal angle. (E-mail: tvarghese@wisc.edu) © 2005 World Federation for Ultrasound in Medicine & Biology.

Key Words: Catheter, Elastography, Elasticity, Elasticity imaging, Intravascular, IVUS, Imaging, Strain, Ultrasound.

INTRODUCTION

Atherosclerosis is a disease of the vessel wall that may occur anywhere in vessels such as in the aorta, carotid or the coronary arteries. Intravascular ultrasound (US), or IVUS, provides real-time high-resolution cross-sectional images of coronary/peripheral arteries, enabling visualization of vessel wall morphology and plaque (Picano et al. 1985; Gussenhoven et al. 1989a, 1989b; Urbani et al. 1993; Wilson et al. 1994; Bridal et al. 1997a, 1997b; Lizzi et al. 1997; Moore et al. 1998; Zhang et al. 1998; Watson et al. 2000). Furthermore, calcified and noncalcified plaque components can be identified. However, the relative sensitivity to identify lipid components remains low and IVUS primarily reveals the geometry of the vessel wall and the presence/absence of plaque. Characterization of the plaque composition in vascular tissue can significantly help in the identification of vulnerable plaque, enabling selection of appropriate interventional techniques to prevent plaque rupture.

Imaging tissue elastic properties has been rapidly developing into a new imaging modality that provides significant new information for diagnosis and treatment (Wilson and Robinson 1982; Krouskop et al. 1987; Parker et al. 1990; Yamakoshi et al. 1990; O'Donnell et al. 1991; Ophir et al. 1991; Varghese et al. 2001). Elastography has been used to characterize strain-based properties of vascular tissue (de Korte et al. 2000a, 2002), de Korte et al. (1998), (2000b) incorporated the 1-D cross-correlation technique (Ophir et al. 1991) for IVUS elastography using different levels of intraluminal pressure to strain the vascular tissue. Local displacement of the tissue is determined using cross-correlation analysis of gated radiofrequency (RF) signals, with the finite differences between successive displacement estimates applied to determine the local strain. IVUS strain is calculated for the vessel wall and plaque and presented as a color-coded ring around the lumen.
vessel-wall boundary. de Korte et al. (2000) have performed both in vitro and in vivo validation studies on excised human coronary and femoral arteries, and Yucatan minipig models with atherosclerosis (de Korte et al. 2002). Initial catheter position correction studies in IVUS elastography was also performed by de Korte et al. (1999), where they derive an expression that corrects for catheter eccentricity and tilt occurring independently within the lumen.

In clinical IVUS practice, the intravascular catheter is inserted into the vessel with the transducer probe located at the tip of the catheter. Unfortunately, the catheter is not always located at the lumen center and, thus, has a significant impact on the resultant IVUS and strain image or elastogram. Hiro et al. (1999) have demonstrated that US signals from plaque depend on the reflection angle and Courtney et al. (2002) showed that both the angle of incidence of the US beam and the distance from the catheter to the region-of-interest have substantial effects on backscattered intensity. Thompson and Wilson (1996) also studied the impact of variations in transducer position and sound speed in IVUS.

In IVUS elastography, misalignment of the US beam and the radial stress introduce errors into strain estimates referred to as “strain projection artifacts” (SPA) (de Korte et al. 1999). Several methods have been used to correct these artifacts, such as the geometric center algorithm (Shapo et al. 1996). Correction factors have also been derived for cases where the catheter is either eccentric (catheter parallel to vessel lumen but not positioned at the center) or tilted (catheter centered but tilted with respect to the vessel longitudinal axes) (de Korte et al. 1999). These expressions show that the measured strain is related to the radial strain by the beam-strain angle, with the derived correction factors applied to correct the SPA inherent under these two specific conditions. However, under general imaging conditions, the catheter will be tilted, eccentric and non-coplanar with respect to the vessel axis, with all of these conditions occurring simultaneously. Therefore, a more general solution that accounts for all of these effects is of interest.

In this paper, we present a general theoretical solution along with correction factors when all of these conditions occur simultaneously during elastographic imaging. The underestimation of radial strain in the presence of the beam-strain angle (angle between the US beam and radial strain) \( \alpha \) is described. Simulation results are used to validate the accuracy of the theoretical correction factors derived in this paper. The simulation results presented enable the clear visualization of the SPA generated with various catheter positions, along with subsequent correction of these artifacts in the strain image or elastogram.

Theory

The theoretical correction factors are derived under the assumption of a cylindrical vessel with a circular cross-section of uniform thickness composed of homogeneous isotropic material. The derivation is obtained using a 3-D xyz coordinate system, where the origin \( O \) denotes the vessel center, and the \( z \)-axis is along the longitudinal direction of the vessel, as illustrated in Fig. 1. The theoretical expression for radial displacement, \( U(r) \), as a function of radius from the center of the vessel, previously derived by Ryan and Foster (1997), is given by:

\[
U(r) = A \cdot r + \frac{B}{r},
\]

where

\[
A = \frac{(1 - 2\nu)(1 + \nu)}{E} \cdot \frac{(p_r - p_0)}{(R_i^2 - R_o^2)} \tag{2a}
\]

\[
B = \frac{(1 + \nu)}{E} \cdot \frac{(p_i - p_0)}{(R_o^2 - R_i^2)} \cdot R_o^2 R_i^2 \tag{2b}
\]

where \( A \) and \( B \) are constants when the vessel dimensions and pressure conditions are as specified below, \( E \) is the circumferential elastic modulus of the vessel wall material, \( \nu \) is the Poisson’s ratio, \( p_i \) is the mean intraluminal pressure over the applied deformation, \( p_0 \) is the mean pressure external to the vessel and \( R_i \) and \( R_o \) are the inner and outer vessel radii at the mean intraluminal pressure of \( p_i \), respectively.

The expression for the radial displacement in eqn (1) can be separated into two terms (i.e., a linear displacement component and a component accounting for the \( 1/r \) stress decay in the displacement). For each radial displacement component, we calculate the impact of
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![Diagram](image)

**Fig. 2.** Eccentric position of catheter from vessel center. Catheter is parallel to the vessel axis.

catheter position errors and then combine both expressions to obtain a composite result. The US beam CA passes through the specified region-of-interest in the lumen wall at point A, which is at a distance \( r \) from the vessel center axis (see Fig. A1 in Appendix A). We denote FA as the radial line from the center of the vessel axis to point A and line FA that is perpendicular to the vessel center axis. To study the impact of catheter position, we first calculate the angle \( \alpha \) between the US beam CA and the radial line FA as a function of azimuthal angle. A component of the derivation was presented by Shi et al. (2003) and the complete derivation is now presented in Appendix A.

**The linear displacement case**

Strain is determined from the gradient of the measured local displacements in the vessel wall, as follows:

\[
\varepsilon = \frac{\delta t_1 - \delta t_2}{\Delta} \quad (3)
\]

where \( \varepsilon \) denotes the strain, and \( \delta t_1 \) and \( \delta t_2 \) are the displacements in the gated windows 1 and 2 separated by a distance \( \Delta \), respectively. The strain projection artifact is first evaluated for the simplest case, where the catheter is eccentric but parallel to the vessel axis. In this situation, as we have shown in Fig. 2, the transducer is positioned at point C, assuming that the US beam is transmitted along the CD direction and that E and F denote the center points of two data segments separated by \( \Delta \) over which the local displacements are computed.

These two points E and F are located at radii \( r_1 \) and \( r_2 \) with respect to the center of the lumen. Estimation of the local strain uses the gradient of the displacement estimates over the two windowed data segments with radii \( r_1 \) and \( r_2 \) as shown in Fig. 2. After the expansion of the lumen caused by changes in the intraluminal blood pressure, points E and F move to \( E' \) and \( F' \), respectively, changing the radii subtended by these points to:

\[
r_1 \to r_1(1 + A) \quad \text{(4a)}
\]
\[
r_2 \to r_2(1 + A). \quad \text{(4b)}
\]

The angle between \( EE' \) and \( FF' \) with CD (US beam) are given by \( \omega_1 \) and \( \omega_2 \), respectively. To calculate the strain, we require the projection of the displacement estimate along the beam direction CD. The projections of the respective displacements are \( [r_1(1 + A) \cos \omega_1 - r_1 \cos \omega] \) and \( [r_2(1 + A) \cos \omega_2 - r_2 \cos \omega_2] \). The projection of the separation (\( \Delta \)) along the US beam (CD) is given by \( r_1 \cos \omega_1 - r_2 \cos \omega_2 \). Using eqn (3), the estimated strain can be written as follows:

\[
\varepsilon = \frac{[r_1(1 + A) \cos \omega_1 - r_1 \cos \omega] - [r_2(1 + A) \cos \omega_2 - r_2 \cos \omega_2]}{r_1 \cos \omega_1 - r_2 \cos \omega_2}. \quad \text{(5)}
\]

Equation (5) illustrates that the strain calculated is independent of the eccentricity of the catheter.

One of the manufacturers of IVUS equipment produces catheters with a built-in tilt angle \( \phi_0 \) between the transducer at the tip and the vessel cross-section, to reduce vibration artifacts in the IVUS image. In this situation, as shown in Fig. 3, the US beam is tilted at an angle \( \phi_0 \) with the vessel cross-sectional plane. The movement of the scatterers within the data segment can be projected along two directions, one along the beam-catheter plane \( A_1 \) and the other perpendicular to the beam-catheter plane \( A_2 \), where the following relationships hold:

\[
A_1 = A \cos \omega,
\]
\[
A_2 = A \sin \omega. \quad \text{(6)}
\]

Because \( A_2 \) is perpendicular to the US beam, it has no contribution to the calculated strain. Similarly, using the derivation from de Korte et al. (1999), the measured displacement is smaller than the real displacement by a factor of \( \cos \phi_0 \) and the distance between the two windowed data segments along the US beam is increased by a factor of \( 1/\cos \phi_0 \). In a similar manner to the derivation of eqn (5), the calculated strain in the presence of a built in tilt angle \( \phi_0 \), can be written as:
Equation (7) illustrates that the calculated strain is scaled by a factor of $\cos^2 \phi_0$ when the catheter has a built-in tilt angle of $\phi_0$.

Let us now consider the case where the transducer is simultaneously tilted and eccentric within the vessel lumen; however, we assume that the catheter and vessel axis are coplanar, as shown in Fig. 4. The projection of the US beam (CD) on the cross-sectional plane is along $CD$, which is at an angle $\omega$ with the point $D'$ to the lumen center. The expansion of the vessel wall can once again be separated into two parts: these are $A r^2 \cos \omega$ along the beam projection direction ($C'D'$) and $A r^2 \sin \omega$ perpendicular to the beam projection direction. The strain calculation is similar to that shown in eqn (7), except that now the angle between $A r^2 \cos \omega$ and the US beam is the beam cross-sectional plane angle $\gamma$. The calculated strain can, therefore, be written as:

$$\varepsilon = \frac{r_A \cos \omega \cos \gamma - r_A \cos \omega \cos \gamma}{\cos \gamma} = A \cos^2 \gamma. \quad (8)$$

where the value of $\cos \gamma$ is given by:

$$\cos \gamma = \frac{\cos \phi_2}{\cos \phi_3}, \quad (9)$$

where $\phi_2$, $\phi_3$ are intermediate angles and are defined in Appendix A. From the derivation above, we show that the correction factor is only related to the angle between the US beam and the cross-sectional plane.

Finally, in the most general case, the catheter and vessel axis are also assumed to be noncoplanar in addition to being eccentric and tilted within the lumen. As shown in Fig. 4, the catheter is at $(a,0,0)$ and the catheter passes through the $y$-$z$ plane at the $(0,b,c)$ point. Note that, in the derivation of eqn (8), we do not limit the catheter to be coplanar with the lumen axis. Instead, we denote the catheter position using the angle $\gamma$ (i.e., $\phi_2$, $\phi_3$), where $\gamma$ is dependent on the catheter coordinates. Therefore, as long as the dependence of the angle $\gamma$ with the rotation of the US beam is known, the correction factor can be easily computed. Thus, eqn (8) is valid for the noncoplanar case, as long as the dependence of the angle $\gamma$ with the rotating US beam can be accurately represented by catheter coordinates $a$, $b$ and $c$.

**The 1/r stress decay displacement case**

For this case, we also start with the simplest situation, where the catheter is eccentric, but parallel to the vessel axis. After expansion of the vessel, the radii subtended by the center points of the two data segments can be written as:

$$r_1 \rightarrow r_1 + \frac{B}{r_1} \quad (10a)$$
$$r_2 \rightarrow r_2 + \frac{B}{r_2} \quad (10b)$$

where $B$ is defined in eqn (1). To calculate the strain, the displacement has to be projected along the beam direction. In a similar manner to that shown in eqn (5), the strain can be calculated using:

$$\text{Fig. 4. Catheter placed in the most general situation in the vessel. Catheter is eccentric, tilted and noncoplanar within the vessel lumen.}$$
Using the relationship
\[ r_1 \sin \omega_1 = r_2 \sin \omega_2 = d_{\text{inc}} \]
and trigonometric relationships, we can simplify the expression as follows:
\[ e = \frac{B \sin \omega_1 \cos \omega_1 \cos \omega_2}{d_{\text{inc}}} \cdot (11) \]

Because the two windowed data segments chosen to calculate strain are very close, \( \omega_1 \approx \omega_2 \) (i.e., \( r_1 \approx r_2 = r \)). The local strain estimate can be written as:
\[ e = -\frac{B}{r} \cos(2\omega). \]  \hspace{1cm} (13)

Similarly, if the catheter has a built-in tilt angle \( \phi_0 \), the calculated strain is given by:
\[ e = -\frac{B}{r} \cos(2\omega) \cos^2 \phi_0. \]  \hspace{1cm} (14)

For the case where the catheter is simultaneously tilted and eccentric within the lumen, the expression can be written as:
\[
\begin{align*}
    e &= -\frac{B}{r} \cos^2 \phi_0 \cos(2\omega) \cos^2 \gamma \\
    &= -\frac{B}{r} \cos^2 \phi_0 \left[ 2 \cos^2 \omega - 1 \right] \cos^2 \gamma \\
    &= -\frac{B}{r} \cos^2 \phi_0 \left[ 2 \cos^2 \alpha - \cos^2 \gamma \right].
\end{align*}
\]  \hspace{1cm} (15)

The derivation of the correction factor for the general noncoplanar case is similar to the linear displacement case described in the previous subsection. As long as the dependence of the angle \( \gamma \) on the rotating US beam is accurately represented by catheter coordinates \( a, b \) and \( c \), eqn (15) remains valid for the general noncoplanar case.

**SIMULATION METHOD AND RESULTS**

**Simulation method**

A simulation program developed in our laboratory by Li and Zagzebski (1999) was used to verify the
validity of the theoretical expression of the correction factor derived in the previous section. The geometry of the cylindrical vessel wall with an inner radius of 4 mm and an outer radius of 10 mm is initially established for the simulation. In the next step, scatterers of a sufficient number to generate Rayleigh statistics are randomly distributed within the vessel wall. This initial distribution of the scatterers is referred to as the precompression state of the simulated phantom. The displacement of the scatterers under intraluminal pressure is then modeled using eqn (2). By setting $B = 0$ and $A = 0$, we simulate the linear displacement and $1/r$ stress decay displacement cases, respectively. The scatterer distributions before and after the application of an intraluminal pressure are saved as corresponding pre- and postcompression cases of the same vessel wall. A simulated US catheter is then placed within the vessel lumen, with the position and orientation of the catheter determined by the catheter position parameters $a$, $b$ and $c$ that are user-defined variables. The values of the catheter coordinates for different simulation conditions are changed by varying the values of the parameters $a$, $b$ and $c$. The transmitted US beam that originates from the catheter propagates to the vessel wall and the corresponding backscattered echoes from scatterers within the vessel wall are received and saved as the radiofrequency (RF) data for pre- and postcompression states of the vessel. For simulations where the catheter incorporates a built-in tilt angle, appropriate tilting of the beam is included in the simulation. Simulation of the IVUS data-acquisition process for generating a cross-sectional image of the vessel is obtained by rotating the US beam along the axis of the catheter.

The simulation program developed by Li and Zagzebski (1999) is based in the frequency domain, where echo signals at each frequency component within an effective frequency band width range of $0 \sim 50$ MHz are simulated. In the final stage of the simulation, we select 20 MHz as the center frequency of the transducer, with a 50% band width. The simulated echo signal spectra over the frequency range from 10 MHz to 30 MHz are summed up, followed by a Fourier transform operation to convert the frequency-domain signal to time-domain signals to obtain RF echo signals. The sampling rate for the simulation was set to 400 MHz and 360 A-line data were

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Fig. 6. Simulation results for the linear displacement case where catheter is only placed in an tilted position. (a) Simulated B-mode image; (b) elastogram before SPA correction; (c) elastogram after SPA correction; (d) strain profile at a 7-mm radial position before SPA correction; and (e) strain profile at a 7-mm radial position after SPA correction.
acquired for each rotation of the catheter about its axis to form the IVUS image. The simulated displacement of the scatterers at the inner radius is 0.5% of the inner radius (i.e., the intraluminal compression is 0.5% at inner radius location). The displacement of scatterers at other radii locations follows eqn (2). For each catheter position, we simulate 25 independent realizations of the pre- and postcompression echo signals. The corresponding elastograms are estimated over these 25 independent realizations and then averaged to obtain a statistical mean and SD of the strain estimates at each pixel position in the elastogram of the vessel. The scatterer positions for each of the independent realizations are randomly distributed, but the scatterer concentration and effective scatterer size are maintained the same for all of the simulations. The effective scatterer size is 50 \mu m in diameter and the scatterer concentration is $10^{10}$ m$^{-3}$.

The US simulation program is written in C++ and runs in a Linux environment. After the completion of each independent simulation, the strain images or elastograms in polar coordinates are calculated using MATLAB (MathWorks, Inc., Natick, MA, USA). The vessel wall boundaries in the elastograms are calculated using eqn (A-9) from Appendix A, and the strain estimates that lie within the inner boundary or lumen and outside the outer boundary (outside the vessel wall) are automatically set to zero. Finally, a graphic user interface program in MATLAB is used to reconstruct the corresponding elastogram in the Cartesian coordinate system.

**Simulation results**

Simulation results mimicking the two cases discussed in the theoretical section, namely the linear displacement model and the stress decay of the displacement, are presented in the following sections.

The linear displacement case. Simulation results in this section present the corresponding IVUS B-mode image and the elastograms before and after correction of the SPA for the known catheter coordinates. These results provide a comparison of the improvement in the strain estimates after correction for the SPA. Each figure

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**Fig. 7. Simulation results for the linear displacement case where catheter is both tilted and eccentric within the lumen.**

(a) Simulated B-mode image; (b) elastogram before SPA correction; (c) elastogram after SPA correction; (d) strain profile at a 7-mm radial position before SPA correction; and (e) strain profile at a 7-mm radial position after SPA correction.
also provides a comparison of the theoretical and simulation results obtained at the center of the vessel wall at 7 mm (note that the lumen has a radius of 4 mm with the outer radius of the vessel set to 10 mm in the simulation).

Figure 5a shows the B-mode image from the simulated vessel when the catheter is placed at an eccentric position 3 mm from the vessel center, but parallel to the vessel axis. In this case, the simulated catheter has a built-in tilt angle of 30°; Fig. 5b shows the corresponding elastogram with the SPA, and Fig. 5c shows the corrected elastogram, with a significant reduction in the SPA. Both the theoretical and simulated strain profiles of the vessel wall before correction of the SPA at a radius of 7 mm are shown in Fig. 5d, where the close correspondence between the theoretical and simulation results are shown. Note that the eccentricity of the catheter does not contribute to SPA, as shown in the elastograms. Figure 5e shows the corrected strain profile of the vessel wall after correction of the SPA at the 7-mm radius.

Generally, the manufacturer’s built-in tilt angle is around 10°. Equation (7) shows that, for the eccentric situation, the calculated strain deviates by a factor of \( \cos^2 \phi_0 \) from the true value, where \( \phi_0 \) is the built-in tilt angle. For catheters with a built-in tilt angle \( \phi_0 = 10° \), the calculated strain is \( \cos^2 10° \), or approximately 0.97 of the true value. In simulations or experiments, it is difficult to separate the strain component because of the built-in tilt angle from the true strain values caused by electronic and other noise artifacts, because the reduction in the true strain is very small. The use of a built-in tilt angle of 30° enables us to verify eqn (7), because it introduces a larger deviation of the true strain values over the errors caused by noise. We, therefore, simulate a manufacturer’s built-in tilt angle of 30° as opposed to the smaller value of 10°. Figure 5b and d denote the elastogram and strain profile at 7 mm, before correction of SPA. The variation in the strain value matches the theoretical prediction. As shown in Fig. 5c and e, the corrected elastogram is uniform and the strain values reach the true applied strain of 0.5% that was simulated. The results shown in Fig. 5c and d confirm the accuracy of eqn (7) in computing the SPA correction.

Figure 8. Simulation results for linear displacement case where catheter is simultaneously tilted, eccentric and noncoplanar (i.e., the most general situation within the lumen). (a) Simulated B-mode image; (b) elastogram before SPA correction; (c) elastogram after SPA correction; (d) strain profile at a 7-mm radial position before SPA correction; and (e) strain profile at a 7-mm radial position after SPA correction.
without a built-in tilt angle, we set $\phi_0 = 0$ degrees and the expression shown in eqn (7) still holds.

Figure 6a illustrates the B-mode image where the catheter is tilted at an angle of 30° with the vessel axis and placed at the center of the vessel. A built-in tilt angle was not simulated for this condition. Because the catheter is tilted, the B-mode image is no longer circular, but ellipsoidal, in shape as seen in Fig. 6a. Figure 6b illustrates the corresponding elastogram with SPA caused by the tilting of the catheter, and Fig. 6c shows the corrected elastogram after reduction of SPA artifacts. In a similar manner as in the previous results, Fig. 6d shows the simulated strain profile and the theoretical strain profile before correction at a radius of 7 mm within the vessel wall. Because of the tilted position of the catheter, the acquired strain value at a specified radius is no longer uniform, although a uniform stress was applied. Note that the estimated strain follows eqn (8), which incorporates SPA caused by the tilted position of the catheter. Reduction of the SPA is easily achieved using the theoretical result obtained with eqn (8) to correct the estimated strain with SPA in Fig. 6b to obtain the corrected elastogram in Fig. 6c. Figure 6e shows the corrected strain profile at a radius of 7 mm.

Figure 7 shows the case where the catheter is simultaneously tilted and eccentric within the vessel lumen. The catheter is placed at an eccentric position 3 mm from the central axis of the lumen and tilted at an angle of 45°. The built-in tilt angle was not simulated for this case. Figure 7a presents the simulated B-mode image, which is similar to that shown in Fig. 6a, where the B-mode image is no longer circular because of the catheter position; Fig. 7b denotes the corresponding estimated elastogram that contains SPA. Note that we can clearly observe the bright and shadowed areas on the elastogram introduced by SPA. The corrected elastogram with the reduced SPA using the theoretical correction in Fig. 7c is fairly uniform. The solid line and error bars in Fig. 7d denote the simulated strain profile and the standard error at a radius of 7 mm before SPA correction; the dotted line is the theoretical prediction of the strain estimates at the 7-mm radial location. The theoretical prediction corresponds very closely to the simulated profile, as shown in Fig. 7d. Finally, Fig. 7e
shows the corresponding strain profiles after correction for the SPA, where the simulated strain profile is now quite uniform.

Figure 8 presents an example of the most general case, where the catheter is also noncoplanar with the vessel axis, in addition to being simultaneously tilted and eccentric within the vessel lumen. In this simulation, we set $a = 3\ mm$, $b = 2\ mm$ and $c = -3\ mm$. In a similar
manner as in the previous simulation, the built-in tilt angle was set to zero. Figure 8a represents the B-mode image and Fig. 8b and c are the corresponding elastograms before and after SPA correction, respectively. The SPA corrected elastogram in Fig. 8c is uniform with a strain value of 0.5%, the applied value of the compression in the simulation. Figure 8d denotes the strain profile at radius of 7 mm location before SPA correction and Fig. 8e shows the corresponding strain profile at the same location after SPA correction. Observe the close correspondence between the strain variations in the theory and simulations showing the strain estimates caused by SPA in Fig. 8d.

The 1/r stress decay displacement case. In a similar manner as in the previous subsection, we also simulate tissue displacement with the 1/r decay in the applied stress with depth. The intraluminal tissue displacement is 0.02 mm at the inner wall or lumen (equivalent to a 0.5% compression of the inner radius) and tissue displacement at other radii undergo a stress decay that follows the 1/r rule. The simplest situation, again, is the case where the catheter is at the lumen center and parallel to the vessel axis. Figure 9a is the B-mode image obtained from a realization of the simulation. Figure 9b shows the stress decay elastogram, where the brightness of the strain estimates in the elastograms decay with depth. Plots of the strain profile with depth over radii of 4 to 10 mm are shown in Fig. 9c. From the strain profile, observe that the strain value decays after a 1/r^2 rule, except for the deviation around the vessel edges, which are artifacts caused by the fact that parts of the windowed data segment lie outside the vessel wall, thereby introducing errors in the strain estimates.

Figure 10 shows the case where the catheter is placed at an eccentric position 3 mm away from the vessel center, but parallel to the vessel axis. The B-mode image is shown in Fig. 10a, and Fig. 10b is the corresponding elastogram before SPA correction. We can clearly visualize the brightness changes along the azimuthal angle direction because of the SPA. Figure 10c shows the SPA-corrected elastogram, where, except for the two bright zones in the region closer to the inner wall or lumen, the strain distribution is very similar to the elastogram depicted in Fig. 9a. The existence of these two bright zones in the elastogram can be explained using Fig. 11 and eqn (15). In eqn (15), the strain value because of the eccentric position of the catheter is proportional to (2cos^2\(\omega\) - 1) = cos2\(\omega\) of the true strain value, where the angle \(\omega\) can be as large as 45° at certain regions, as shown in Fig. 11. So, the resultant strain estimate would be close to zero; however, this is very unlikely because of noise artifacts in the elastogram. Nevertheless, these strain estimates would be quite small. Now, when SPA are corrected for in the elastogram, strain estimates are divided by the correction factor obtained using eqn (10), which amplifies the noise in these regions of the corrected elastogram.

Figure 10d shows the strain profile at a radial location of 7 mm before SPA correction, where we observe a close correspondence between the simulation and theoretical prediction. Figure 10e shows the corresponding strain profile after correction and Fig. 10f shows the strain profile along the radii from 4 mm to 10 mm. Observe that the strain profile follows the 1/r^2 decay curve well, except near the lumen or initial portion of the vessel wall, which is because of the amplified noise artifacts.

Figure 12 shows the situation where the catheter is at the center of the vessel, but tilted at 30° from the vessel axis. Figure 12a denotes the B-mode image and Fig. 12b is the elastogram before SPA correction. Because of the catheter position, the B-mode image and elastogram are no longer circular, but elliptical, as previously described. Figure 12c denotes the elastogram after SPA correction. Note that Fig. 12b and c are similar because the variance of the SPA is quite small. We can also clearly observe this aspect in Fig. 12d, which shows the strain profile before correction at a 7-mm radial location. The figure indicates a strain variation along the azimuthal angle and the theoretical curve is able to predict the simulation results obtained. Finally, Fig. 12e shows the strain profile after correction at the same location.

Figure 13 shows the case where the catheter is simultaneously eccentric (3 mm from the vessel center) and tilted at 45°. Figure 13a represents the B-mode image and Fig. 13b is the elastogram before SPA correction. Again, because of the position of the catheter, the B-mode image is no longer circular, but ellipsoidal,
in shape. The elastogram in Fig. 13b is quite dark because of the SPA artifacts, whereas, in Fig. 13c, the elastogram after correction, the whole image is bright and the $1/r^2$ trend is obvious. We can also clearly visualize the two brighter regions closer to the inner wall because of the amplification of noise artifacts. Figure 13d shows the strain profile before correction at a 7-mm radial position, which indicates a large strain variation along the azimuthal angle, with the theoretical curve providing a good prediction of the simulation result as well. Figure 13e shows the strain profile after correction of the SPA at the same location.

Finally, Fig. 14 provides an example of the most general case (i.e., where the catheter is also noncoplanar with the vessel axis in addition to being simultaneously eccentric and tilted). For this simulation, we set $a = 3$ mm, $b = 2$ mm and $c = -3$ mm. Figure 14a again is the B-mode image and Fig. 14b shows the elastogram before SPA correction. In a similar manner as shown in previous simulation examples; the B-mode image and elastogram are not circular in shape. Figure 14c is the elastogram after correction of SPA artifacts. In Figure 14d, once again we can clearly observe the variation in the simulated strain profile predicted by the theoretical curve. Figure 14e shows the strain profile after correction at the same location.

**DISCUSSION**

Knowledge of plaque composition, especially lipid-rich and mixed (fibrous, lipid, calcified), and plaque vulnerability through its elastic properties can significantly assist the surgeon in selecting appropriate interventional techniques. In this study, we derive a general solution for catheter position correction for intravascular strain estimation. This general solution covers the entire gamut of possible catheter positions within the vessel lumen (i.e., eccentric, tilted and noncoplanar). As shown in the paper, these situations can exist independent of each other or could exist
simultaneously. However, the general solution presented provides a solution for all possible catheter positions, either occurring independently of each other or simultaneously. The correction for the SPA is only dependent on the catheter position in the vessel, which has been verified using simulations. The results obtained at various conditions all match the theoretical predictions. Previous theoretical expressions for SPA solutions by de Korte et al. (1999) deal only with cases where the catheter position is either tilted or eccentric, with neither condition occurring simultaneously. In addition, they derived separate expressions to deal with each condition. In this paper, we present a more general situation (i.e., the tilted, eccentric and nonco-planar situations that may occur either independently of each other or simultaneously), and the general solution is obtained using a single expression.

Generally because of the flexible nature of the catheter, it is difficult for the clinician to accurately position the catheter within the vessel lumen. The cross-sectional US image from the IVUS clinical scanner can help the clinician visualize the plaque; however, it is difficult to determine the catheter position parameters \(a, b\) and \(c\) from the IVUS image itself. Other imaging modalities (such as x-ray angiography) or other independent tracking techniques have to be used to determine the catheter position.

The blood vessel model used in this paper assumes the vessel to be an infinite length circular cylinder, where the center of the inner wall and outer vessel wall are the same. However, under practical conditions for \textit{in vivo} imaging, this is usually not the case. The noncircular shape of the vessel will introduce errors in the theoretical expression derived in this paper, under the conditions where the expansion of the vessel wall tissue under intraluminal pressure may not be only along the radial direction, introducing errors in both the strain estimation algorithm and the theoretical correction factor and, even with a fixed catheter position, the catheter position parameters \((a, b\) and \(c)\) will vary for different locations within the vessel because of its curvature. This artifact could be overcome by...
acquiring several adjacent points on the vessel wall to determine an estimate of the equivalent local radius of the vessel.

The general solution presented in this paper is an expression that includes the azimuthal angle $\theta$, where $\theta$ is defined as the angle between the x-axis and the intersection line of the US beam-catheter plane and the x-y plane. This definition of the azimuthal angle is, therefore, around the transducer $C$, instead of the usual definition of the angle around the vessel center $O$, as shown in Fig. A1 in Appendix A. This scheme is convenient for our simulation results, because the beam originates from the transducer and sweeps in 360° arcs around the transducer. In the theoretical expression and simulation results, we can easily set the initial $\theta = 0$ position at the x-axis and increase the $\theta$ values counter-clockwise from the z-axis. However, under practical conditions, the situation may be more complicated because it is difficult to pinpoint the exact azimuthal direction of the beam with our experimental data-acquisition system on a single RF data frame. In the meantime, the catheter could vibrate and rotate off its previous center of mass during different sweep cycles (Thompson and Wilson 1996) used to generate the real-time IVUS images. All of the above-described conditions make SPA correction more challenging.

Because many factors, such as the actual transducer dimensions, catheter vibration effects, alignment of pre- and postcompression RF data, homogeneity of tissue etc., affect the theoretical correction expression in clinical practice, with some of these being more critical than the possible errors in the catheter position parameters ($a$, $b$ and $c$), we did not derive the error propagation formula because of the uncertainties in the catheter-position parameters in this paper. The mathematical derivation is possible; however, the final expression obtained is fairly complicated. Because we derive the theoretical closed-form solutions for the correction factors in this paper, errors caused by incorrect estimations of the parameters $a$, $b$ and $c$ can be evaluated by varying the parameters $a$, $b$ and $c$ in the corresponding theoretical cor-

Fig. 14. Simulation results for nonlinear $1/r$ stress decay displacement case. Catheter is placed in the most general case (i.e., simultaneously eccentric, tilted and noncoplanar within the lumen). (a) Simulated B-mode image; (b) elastogram before SPA correction; (c) elastogram after SPA correction; (d) strain profile at a 7-mm radial position before SPA correction; and (e) strain profile at a 7-mm radial position after SPA correction.
rection expressions and obtaining the difference between the theoretical correction values. Interested readers are referred to Shi et al. (2003), where we have demonstrated the error-propagation approach discussed in this paragraph.

In the derivation of the theoretical correction factor, we separate eqn (1) into two parts and derive the correction factor for both the linear displacement and $l/r$ stress decay in the displacement. For general Poisson’s ratio material, the final correction factor is the weighted summation of the correction factor for the linear-displacement case and the displacement $l/r$ decay case, as shown in eqn (1). For incompressible materials with Poisson’s ratio $\sim 0.5$, the linear displacement part disappears and only the displacement $l/r$ decay case decides the correction factor.

During the calculation of elastograms from the simulated RF data, we noticed that the window length selection also affects the noise characteristics of the elastogram. For the linear displacement case, we use a 0.03-mm window length and, for the $l/r$ stress decay displacement case, we evaluated several window lengths and observed that the combination of a 0.05-mm window length and up-sampling of the RF data to a sampling rate of 400 MHz provides the best image quality. Because, in the linear-displacement case, the displacement increases at the larger radii locations, smaller window lengths sometimes introduce decorrelation at larger radii. On the contrary, for the $l/r$ stress-decay displacement case, displacement decreases dramatically with an increase in the radii, where the smaller window lengths perform better.

The boundaries of all the elastograms were calculated using eqn (A-9) in Appendix A, where $y$ values at the inner radius denote the inner boundary and the $y$ values at the outer radius denote the outer boundary. The resultant boundaries are smooth and the entire computation is performed automatically, which also implies the accuracy of our theoretical derivation.

Figures 10c, 13c and 14c also show the artifacts introduced because the correction factors are close to zero. Generally, cases where the theoretical correction factors become close to zero occur in locations close to the inner wall of the vessel, which is also generally the plaque location. Removal or reduction of these artifacts is difficult. One solution would be the use of a small regularization factor. However, these artifacts occur under specific circumstances, such as with specific combinations of inner radius and catheter position parameters ($a$, $b$ and $c$). This implies that, for IVUS elastography plaque characterization, one can adjust the catheter position inside the vessel (i.e., by moving the catheter away from the plaque in such cases) to avoid correction factors close to zero at regions that contain plaque.

**CONCLUSION**

We have derived a closed-form theoretical expression that provides for the correction of the SPA when the catheter is simultaneously tilted, eccentric and noncoplanar within the vessel lumen. A frequency-domain simulation program is used to verify and validate the theoretical results for different practical imaging conditions. The simulation results obtained demonstrate a close correspondence to the theoretical derivation.

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**REFERENCES**


\[ \delta = \text{The angle between the catheter and the intersection line of the US beam-catheter plane and x-y plane} \]

\[ \phi_0 = \text{The manufacturer’s built-in tilt angle, the angle between transducer and catheter; if } \phi_0 = 0, \text{ then the manufacturer’s built-in tilt angle is zero} \]

\[ \phi_2 = \text{The angle between the US beam and intersection line of the US beam-catheter plane and x-y plane} \]

\[ \phi_3 = \text{The angle between the intersection line of the US beam-catheter plane and x-y plane, the line that connects the transducer and the projection of the point where US beam reaches the vessel wall on the x-y plane} \]

\[ \gamma = \text{The angle between the US beam and the x-y plane} \]

\[ \theta = \text{The azimuthal angle, defined as the angle between the x-axis and the intersection of the US beam-catheter plane and the x-y plane} \]

\[ \omega = \text{The angle between the projection line of the US beam on w-y plane and the radial line that passes through the same point on the vessel wall as the projection} \]

\[ \xi = \text{Eccentricity, defined as } a/r \]

**APPENDIX A**

The geometric relationship between the catheter and coordinate system is shown in Fig. A1. For a vessel with a circular cross-section of radius \( r \), we assume that the transducer is positioned at point C on the x-axis with coordinates of \((a,0,0)\), where \( a/r \) denotes the eccentricity of the transducer. The catheter crosses the y-z plane at point P \((0,b,c)\). IVUS transducers are typically designed to subtend an angle of \( \phi_0 \) with the vessel wall to avoid mechanical vibration. The catheter-

\[ a = \text{The eccentric position of the transducer on x-axis} \]

\[ b = \text{The y-coordinate of the point where catheter passes through the y-z plane} \]

\[ c = \text{The z-coordinate of the point that catheter passes through the y-z plane} \]

\[ h = \text{The distance between the point where US beam reaches the vessel wall to the x-y plane} \]

\[ r = \text{The radius of the vessel lumen} \]

\[ y = \text{The distance between the transducer and vessel wall} \]

\[ \alpha = \text{Strain beam angle, the angle between US beam and radial strain} \]

\[ \beta = \text{The angle between the US-catheter plane and the x-y plane} \]

**NOTATION OF VARIABLES**

\[ \frac{a}{r} \text{ represents the catheter. This shows the variables used in derivation of the closed form expression.} \]

Fig. A1. Geometric relationship between the catheter and coordinate system. The z-axis denotes the axis of the vessel, the x-y plane is the cross-sectional plane of the vessel, and P-C represents the catheter. This shows the variables used in derivation of the closed form expression.

\[ \frac{a}{r} \]
beam plane denoted by PCA now crosses the x-y plane with angle \( \beta \) along the line CE. The azimuthal angle \( \theta \) is defined as the angle between the x-axis and the line CE. When the catheter rotates along PC, the US beam CA scans a conical swath along PC. The US pulse reaches the vessel at point A, where A satisfies the condition that the projection of A on the x-y plane (defined as B) has a distance \( r \) from the origin point O. Similarly, we can also draw AF/BO and point F (with coordinates \((0,0,b)\) on the z-axis). Note also that AF is \( r \), the radius of the vessel. The angle \( \alpha \) between US beam CA and radial line AF is the strain-beam angle. The relationships derived in the following six intermediate equations are necessary to determine the geometric relationship between \( \alpha \) and \( \theta \).

**Intermediate equation I**

We denote the propagation path of the US beam CA from the transducer to the vessel wall as \( y \), the angle between BC and CE as \( \phi_3 \), and the angle between AC and CE as \( \phi_2 \), as shown in Fig. A1, we obtain:

\[
\phi_3 = \arccos \frac{y \cdot \cos \phi_2}{\sqrt{y^2 - h^2}} \tag{A-1}
\]

Equation (A-1) is the same as eqn (B-4) in Appendix B.

**Intermediate equation II**

From triangle OBC in Fig. 2, when \( \theta < \pi \), the \( \angle OCB = \pi - \theta + \phi_3 \) and, for \( \theta > \pi \), \( \angle OCB = \phi_3 - \pi + \theta \), so:

\[
r^2 = y^2 - h^2 + a^2 - 2a \sqrt{y^2 - h^2} \cos (\angle OCB) \tag{A-2a}
\]

\[
r^2 = a^2 + (y^2 - h^2) + 2a \sqrt{y^2 - h^2} \cos (\pi - \phi_3) \tag{A-2b}
\]

**Intermediate equation III**

For the triangles ABE and ACE, we have \( AB \perp BE, AE \perp EC \), so:

\[
h = y \sin \phi_2 \sin \beta \tag{A-3}
\]

**Intermediate equation IV**

To find the angle \( \beta \) between the catheter-beam plane ACP in Fig. A1 and the x-y plane, we extend PC to point D in Fig. A2, with PC = CD. We then project point D on the x-y plane at J. We can prove that triangle CJD is exactly the same as triangle CQP. From J, we then draw a perpendicular line with respect to line CE and the intersection point is G. If the angle JGD is \( \beta \), then:

\[
\tan \beta = \frac{c}{\sqrt{a^2 + b^2} \cdot \sin \left( \theta + \arctan \frac{b}{a} \right)} \tag{A-4}
\]

**Intermediate equation V**

We also require a relationship between the angle \( \phi_3 \) and the azimuthal angle \( \theta \), as illustrated in Fig. A3. For convenience, we extend the line CE to point H and set CH = \( a \); the angle \( \phi_3 \) can be written as:

\[
\phi_3 = \phi_0 + \pi - \arccos \frac{b \sin \theta - a \cos \theta}{\sqrt{a^2 + b^2 + c^2}} \tag{A-5}
\]

Equation (A-5) is the same as the eqn (C-4) in Appendix C.

**Intermediate equation VI**

And, finally, from triangle ACF, we have:

\[
a^2 + h^2 = y^2 + r^2 - 2yr \cos \alpha \tag{A-6}
\]

So far, we have six independent eqns (A-1)–(A-6) and six independent variables \( h, y, \alpha, \beta, \phi_0 \) and \( \phi_3 \) that are used to obtain a closed form solution for the beam-strain angle \( \alpha \).

From eqns (A-3) and (A-4), we have:

\[
h = y \sin \phi_2 \cos \alpha \tag{A-7}
\]

Substituting eqns (A-1) and (A-7) into eqn (A-2), we obtain:

\[
y^2 \left[ 1 - \frac{c^2 \sin^2 \phi_2}{c^2 + (a^2 + b^2) \sin^2 \left( \theta + \arctan \frac{b}{a} \right)} \right] + y \cdot 2a \left[ \frac{\cos \theta \cos \phi_2}{\sqrt{a^2 + b^2} \sin \left( \theta + \arctan \frac{b}{a} \right)} \right] + \sin \phi_2 \frac{\sqrt{a^2 + b^2} \sin \left( \theta + \arctan \frac{b}{a} \right)}{\sqrt{c^2 + (a^2 + b^2) \sin^2 \left( \theta + \arctan \frac{b}{a} \right)}} + (a^2 - r^2) = 0. \tag{A-8}
\]

We solve this equation for \( y \) and obtain:

\[
y = -n + \frac{\sqrt{n^2 - 4mk}}{2m}. \tag{A-9}
\]

where

\[
m = \left[ 1 - \frac{c^2 \sin^2 \phi_2}{c^2 + (a^2 + b^2) \sin^2 \left( \theta + \arctan \frac{b}{a} \right)} \right]. \tag{A-10a}
\]

Fig. A2. Geometric relationship between catheter and coordinate system used in the derivation of intermediate equation IV.

Fig. A3. Geometric relationship between catheter and coordinate system used in the derivation of intermediate equation V.
From eqns (A-6) and (A-7), we obtain:

\[
\alpha(\theta) = \arccos\left(\frac{1}{2\sqrt{2}} \left( r^2 - a^2 \right) + \frac{c^2 \sin^2 \phi_2}{c^2 + (a^2 + b^2) \sin^2 \left( \theta + \arctan \frac{b}{a} \right)} \right) \tag{A-11}
\]

where \( y \) and \( \phi_2 \) are given by eqns (A-9) and (A-5), respectively, which are functions of \( \theta \). Substituting into the expression above, we get an expression for the beam-strain angle \( \alpha \) as a function of the azimuthal angle \( \theta \).

We now consider a special case of eqn (A-11). To evaluate the situation when the catheter is coplanar with the vessel axis, we set \( b = 0 \) and define the tilt angle as:

\[
\phi_1 = \arctan \frac{a}{c}. \tag{A-12}
\]

Then, eqn (A-4) can be simplified to:

\[
\tan \beta = \frac{c \tan \phi_1}{\sin \theta}. \tag{A-13}
\]

Similarly, eqn (A-5) can be simplified to:

\[
\phi_2 = \phi_0 + \frac{\pi}{2} - \arccos(-\sin \phi_1 \cos \theta) = \phi_0 - \frac{\pi}{2} + \arccos(\sin \phi_1 \cos \theta). \tag{A-14}
\]

In a similar manner, eqn (A-10) can be simplified to:

\[
m = \left[ 1 - \frac{c^2 \sin^2 \phi_1}{c^2 + a^2 \sin^2 \theta} \right] = (\sin^2 \theta + \cos^2 \phi_2 \tan^2 \phi_1 + \tan^2 \phi_1). \tag{A-15a}
\]

\[
n = 2a \left[ \tan \phi_2 + \sin^2 \phi_1 \tan \phi_2 \sqrt{\frac{a}{c^2 + a^2 \sin^2 \theta}} \right] = 2a (\cos \theta \cos \phi_2 + \sin \phi_1 \sin \phi_2) \tag{A-15b}
\]

\[
k = (a^2 - r^2) \tag{A-15c}
\]

In addition, eqn (A-11) is reduced to:

\[
\alpha(\theta) = \arccos\left(\frac{1}{2\sqrt{2}} \left( r^2 - a^2 \right) + \frac{c^2 \sin^2 \phi_2}{c^2 + (a^2 + b^2) \sin^2 \theta} \right) \tag{A-16}
\]

The eqns (A-14) to (A-16) have the same notation as in our previous work (Shi et al. 2003) and, therefore, the solution is similar. These equations can also be simplified to obtain the de Korte et al. (1999) solution as shown in Shi et al. (2003).

**APPENDIX B**

For the triangular prism ABCE in Fig. A1, we have \( AB \perp BC \), \( BE \perp EC \) and \( AE \perp EC \). Denoting \( AC = y \) and \( AB = h \) and because

\[
AB \perp BC, \quad BC = \sqrt{y^2 - h^2}. \tag{B-1}
\]

For triangle BEC, in Fig. A1, we have:

\[
\cos \phi_1 = \frac{EC}{BC}. \tag{B-2}
\]

In a similar manner for triangle AEC in Fig. A1, we obtain:

\[
\cos \phi_2 = \frac{AC}{EC}. \tag{B-3a}
\]

\[
EC = AC \cos \phi_2. \tag{B-3b}
\]

Substituting for EC from eqn (B-3) into eqn (B-2), we obtain an expression for angle \( \phi_1 \) as follows:

\[
\phi_1 = \arccos \frac{y \cos \phi_2}{\sqrt{y^2 - h^2}}. \tag{B-4a}
\]

\[
\phi_2 = \arctan \frac{y}{\sqrt{y^2 - h^2}}. \tag{B-4b}
\]

**APPENDIX C**

In Fig A3, \( OP = \sqrt{b^2 + c^2} \), \( CP = \sqrt{a^2 + b^2 + c^2} \) and \( OH = a\sqrt{2} + 2\cos \theta \) if we define \( HQ = w \) and \( HP = u \) and the angle COH is \( \sigma \). Then, because triangle HCO is symmetrical, we can calculate:

\[
\cos \sigma = \frac{1}{2} \sqrt{2 + 2\cos \theta} \tag{C-1}
\]

We also obtain:

\[
w^2 = a^2 + b^2 - 2ab \sqrt{2 + 2\cos \theta} \cos \left( \frac{\pi}{2} - \sigma \right) = a^2 + 2b \cos \theta + b^2 - 2ab \sin \theta \tag{C-2}
\]

And we can find \( u^2 = c^2 + w^2 \). In triangle CHP, we have \( \delta = \pi - \phi_2 - (\pi/2 - \phi_0) \) and:

\[
u^2 = a^2 + (a^2 + b^2 + c^2) - 2a \sqrt{a^2 + b^2 + c^2} \cos \delta \Rightarrow \cos \delta = \frac{b \sin \theta - a \cos \theta}{\sqrt{a^2 + b^2 + c^2}} \tag{C-3}
\]

And, thus, we obtain:

\[
\phi_2 = \phi_0 + \frac{\pi}{2} - \arccos \frac{b \sin \theta - a \cos \theta}{\sqrt{a^2 + b^2 + c^2}}. \tag{C-4}
\]