1. In this problem, we explore the properties of the Hilbert transform further.

A. Create a 1 kHz zero-mean cosine (voltage) waveform with amplitude 2 V containing at least two complete cycles. Use a sufficiently large number of samples in the digital waveform that sampling is not an issue in the analysis.

B. Compute the discrete Fourier transform of that waveform. What are the units of the magnitude, phase and frequency? Plot the waveform and its FFT with appropriate labeling for the axes.

2. In class we discussed restoration of a signal both by direct convolution with a sinc waveform and by “zero-padding” in Fourier transform space.

A. Explore interpolation using zero-padding of the DFT. Let $F_4(k) = \{1 1 1 1\}$, $F_8(k) = \{1 1 1 1 0 0 0 0\}$, and $F_{16}(k) = \{1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0\}$, all generated by sampling the 1D image $\text{rect}[(\zeta-2.5)/4]$ at intervals $\Delta \zeta = 1 \text{ mm}^{-1}$.

B. Apply ifft() in Matlab to all three, compare the three results, and reconcile any differences with the continuous inverse Fourier transform: $f(x) = 4 \exp(-i5\pi x) \text{sinc}(4x)$. Note, again, that Matlab does not follow the usual scaling convention for the DFT. The fft() function does not scale by 1/N for the forward transform, but ifft(), the inverse transform, does scale by 1/N.

3. Consider the object $f(x,y)$ shown in Fig. 1a, a square centered at the origin with length 2 units per side with signal value B. Outside the square is zero signal. Also consider the point spread function $h(x,y)$ in Fig. 1b, consisting of a delta function at the origin and 1/4 amplitude delta functions located at $(0,\pm2)$ and $(\pm2,0)$.

A. What is the output $g(x,y)$ if $f(x,y)$ is input to a transformation having the PSF of $h(x,y)$?

B. Next consider the array of shaded squares in Fig. 1c. Each individual square is identical to the one in Fig. 1a and the array extends indefinitely in the $\pm x$ and $\pm y$ directions. The blank areas between the shaded squares have zero signal. What is the output if this array is input to the same transformation as in Part A?
4. Usually “quadrature” detection is used in the nuclear magnetic resonance signal detection process. A diagram of the circuitry is shown in Figure 2. Two detector antennas measure the complex signal from the rotating transverse magnetization. These are added together, amplified, and sent along a single channel from the scanner to the control room. There the summed signal is sent to two separate channels, each is demodulated, low-pass filtered, and digitized.

Figure 1:

A. In the rotating frame suppose the "in-phase" portion of $M_{xy}$ is $m_I(t)$ and the "quadrature" phase is $m_Q(t)$. Then the signals $s_I(t)$ and $s_Q(t)$ measured by the I and Q detectors are:

$$s_I(t) = m_I(t) \cos(2\pi f_L t)$$

$$s_Q(t) = m_Q(t) \sin(2\pi f_L t)$$

where $f_L$ is the Larmor frequency, or the “carrier” frequency of the signal. Here we show that the final signals $y_I(t)$ and $y_Q(t)$ can equal the original $m_I(t)$ and $m_Q(t)$. Start by writing the explicit expression for the sum signal: $s(t) = s_I(t) + s_Q(t)$. This signal is transmitted from the scanner to the control room electronics.

B. Next demodulate $s(t)$ through each channel. That is, multiply $s(t)$ by $\cos(2\pi f_L t)$ and simplify. Repeat for $\sin(2\pi f_L t)$. Hint: Make judicious use of trig identities. The resultant signals $x_I(t)$ and $x_Q(t)$ are the inputs to the low-pass filters.
C. Suppose the Fourier transform of $m_I(t)$ and $m_Q(t)$ are shown in Fig. 2. Suppose that each is bandlimited with $M(f) = 0$ for $|f| > f_c$ and moreover assume that $f_c << f_L$. Draw the Fourier transforms, $X_I(f)$ and $X_Q(f)$, of $x_I(t)$ and $x_Q(t)$, respectively using your results from Part B and appropriate Fourier transform theorems.

D. Write an expression for the necessary response of the low pass filters in order for $y_I(t)$ and $y_Q(t)$ to exactly match $m_I(t)$ and $m_Q(t)$. 