Short answer and essay:

5 pts each

1. (5 pts) How many degrees of freedom are there in a 3D non-deformable rigid body transform?

   \[ \text{3D rigid body implies 3 degrees of freedom (dofs) for translation} + 3 \text{ dofs for rotation} = 6 \text{ total dofs} \]

2. (5 pts) The affine transform preserves parallel lines. What are the three major types of displacement that are allowed under this constraint?

   \[ \begin{align*}
   \begin{array}{c}
   \text{translation} \\
   \text{shear} \neq 0
   \end{array}
   \end{align*} \]

   \[ \begin{align*}
   \begin{array}{c}
   \text{rotation} \neq 0
   \end{array}
   \end{align*} \]

3. (10 pts) Automated registration methods require a cost function with a global maximum or minimum. What are some of the numerical methods used to find the global maximum? Describe generally how they work.

   Gradient ascent (or descent) methods are most commonly used when the cost function \( f(x, y) \) represents some surface on \( (x, y) \) that is generally either a convex or concave function.

   The gradient, \[ \nabla f(x, y) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \] can be used in an iterative manner to find the point where \( \nabla f(x, y) = 0 \) within some error, \( \epsilon \). In practice noise makes \( f(x, y) \) only approximately convex or concave. To avoid local minima, simplex search kernels are often used that allow reflection, expansion, and contraction operations.
4. (20 pts) Name at least 2 different cost functions used in image registration and non-linear curve fitting. Briefly discuss the underlying rationale for the use of the respective cost functions.

1. Least squares is the most ubiquitous cost function and it is based on the concept of maximum likelihood, which assumes that the pdf of the underlying uncertainty in the registration is described by a Gaussian function.

2. Normalized mutual information is another common cost function. It is based on the entropy measure of image information. 
\[ E(A) = \frac{L}{i=1} \sum \ln[p_i(A)] \ln[p_i(A)] \]
\[ E(A, B) = \frac{1}{L} \sum \ln[p_{i}(A, B)] \ln[p_{i}(A, B)] = \text{the joint entropy of } A \land B \]

Maximize:
\[ I(A; B) = E(A) + E(B) - E(A, B) \]

Functional imaging includes a wide range of methods used to assess physiological processes. Respond as to whether the following statements regarding functional imaging are true or false (8 pts each):

5. Perfusion measurement in medical imaging applications requires radioactive tracers in order to provide sufficient temporal and spatial resolution. 

6. Injected contrast agents can be safely used in healthy volunteers and patients in all circumstances.

7. A rapid intravenous injection of a tracer is the ideal approach for estimating perfusion using the linear systems model.

8. Functional imaging only applies to imaging in the brain.

9. Different types of radio-labeled tracers are characterized by different kinetics, including freely diffusible, trapped, and receptor binding processes.
10. (20 pts) Consider an image application with a time-dependent signal process you think might be well modeled by the function, \( f(t) \), below:

\[
f(t) = At^ae^{-\alpha t},
\]

where \( t \) is the continuous time variable, and \( A \) and \( \alpha \) are constants. Discuss two approaches to fitting this model to the measured data using mathematics where appropriate (although rigorous proofs are not required). What is the cost function used in each case and how do the two approaches fundamentally differ.

1. The function can be log-linearized using \( \ln(\cdot) \).

\[
\ln f = \ln A + \alpha t + \ln t - \alpha t.
\]

The least squares cost function \( (\chi^2) \) can now be minimized s.t.:

\[
\frac{\partial \chi^2}{\partial a_k} = \frac{2}{\sigma_i} \sum_{i=1}^{m} \left[ \frac{1}{\sigma_i} \left[ (a_1 + a_2 \ln t - a_3 t) - f_i \right] \right] = 0
\]

2. The second approach is to use a non-linear fitting algorithm based on a least squares cost function that is not known analytically. A simplex search method using the steepest descent gradient approach as described in the answer to problem 3 can be used.